Desk Book.

Bethlehem School.

PRIMARY ARITHMETIC

FIRST YEAR

FOR THE USE OF TEACHERS

BY

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PREFACE.

THIS book is one of a series soon to be issued. The point of view from which it is written is indicated in the introduction.

The essence of the theory of teaching arithmetic can be expressed in a few sentences. The fundamental thing is to induce judgments of relative magnitude. The presentation regards the fact that it is the *relation* of things that makes them what they are. The *one* of mathematics is not an individual, separated from all else, but the union of two like impressions: the *relation* of two equal magnitudes. A child does not perceive this *one* until he sees the *equality* of two magnitudes. He will not become sensitive to relations of equality by handling equal units with the attention directed to something else, as the color, the texture, or the how many; nor by one or two experiences in comparing magnitudes.

To aid the learner in seeing a 1 as the *relation* of two equal units, a 2 as the *relation* of a unit to another one half as large, a $\frac{1}{2}$ as the *relation* of a unit to another twice as large, we must induce the repeated acts of comparing which bring these relations vividly before the mind. With this purpose the child is not required to build out of parts a whole which he has never seen, nor expected to discover a relation in the absence of one of the related terms. He does not begin with elements. He is not prevented from

seeing things as they are by pushing elements into the foreground. The mind grasps something vaguely as a whole, moves from this to the parts, and gradually advances to a clearer and fuller idea of the whole. Whether the object of study be a flower, a picture, a cubic foot, or a six, the process of learning is the same. If we promote progress in the discovery of relations of magnitude, we will make it possible for the compared wholes to be pictured in their full extent, thus affording opportunity for comparing, for activity in judging. There is no such opportunity when a child who has no idea of a thing constructs it mechanically from given parts. Creation, in any subject, requires a basis in elementary ideas.

It is not to be forgotten that there is a wide difference between seeing that the relation of two particular things is 8, and realizing 8 as a relation, realizing it in such a way that it can be freely used without misapplying it.

There is no real progress unless the mind is gradually gaining power to think of things not present to sense, and to think of a relation apart from a particular thing. But there is no way to promote this progress except by securing continued activity of sense and mind. The child grows into the idea 8, slowly and unconsciously but surely, under right conditions. A cube does not become known by counting surfaces, edges, etc., again and again, but by observing other forms and many different cubes. Through repeated acts of dissociating and relating, what is particular sinks out of sight and the common trait stands out. This principle is of general application.

There should be constant calls for reperception, for judging and verifying. Only by multiplying experiences in the concrete, by noting the same relation in many different things and in many different conditions, does the child come to know a relation as it is.

The slow development of the power to form perfectly quantitative judgments is considered. Hence the earlier work makes no demand for close analysis. It provides for a gradual advance toward exactness. The exercises are only suggestive. The condition of the child determines what he should do. But in any case, the work in the beginning should be so simple that it can be done easily; it should look to the free action of both body and mind.

The child interested in finding colors and forms wishes to move about, to touch and handle things. Out of school he combines thinking and acting. Why should he not do so in the school? Interest will lead the child to control himself, but repression from without induces dullness, indifference, and antagonism. Force a child to preserve a regulation attitude, to keep his nerves tense, and you destroy the foundation of healthful mental activity. In the transition from home to school life, careful provision should be made for the *whole* child to express himself.

Attention is asked to the remarks upon over-direction, premature questioning, demands for analysis beyond the inclination and power of the pupil and for outward forms which are not the genuine expression of the child.

Great importance is attached to that order of work which puts things before the pupil and leaves him free to see and to tell all he can before interfering with his action by questioning or direction. Questions have their uses. They serve to arouse attention, to aid in testing the pupil's view, and may lead to the correct use of new forms of expression. But there are effects of questioning which are too often overlooked. Questions do for the child what he should do for himself; they conceal his attitude toward the work and prevent your seeing what he would do unaided. They call attention to details for which the mind may not be prepared, and present a partial, fragmentary view. The

questioning may be logical, but the learner connects only that which he himself relates. Questions cause the teacher to suppose that the child grasps what is not appreciable by him, and so prevent the adaptation of the work. To attempt to force through questions what *you* see in a poem, picture, or problem, instead of leaving the pupil to discover what *he* is prepared to see, is to ignore the true basis of advance, to disregard the law that the mind passes from vague ideas to those fuller and more exact, only through its own acts of analysis and synthesis. Free work reveals the pupil and makes it possible to meet his needs.

This view furnishes no excuse for random, desultory work. The teacher must carefully select the means, whether the ideas into which he wishes to lead the child are mathematical, biological, or historical.

In conclusion it is urged that any success is dangerous which lessens the susceptibility of the mind to new impressions. We may be so successful in training the child to reproduce as to destroy his power to produce. Progress is impossible without growing power to do unconsciously what was at first done consciously; but accuracy is not to be desired at the expense of growth. The purpose of automatic action in education is not to restrict, but to set force free. When the work of the school is mechanical it weakens the relating power, the power to act in new circumstances, and thus lowers the child in the scale of being.

As insight into the subject and contact with the child enable us to open right channels for free action, there will be little occasion for drills. The fresh, vigorous effort of involuntary attention carries the child forward with surprising rapidity. Out of *self*-activity comes the self-control which gives strength to persist.

THEORY OF ARITHMETIC.

INTRODUCTION.

The following quotations may be found suggestive of working ideals. The teacher who enters into their spirit will feel the need of knowing both the child and the subject. She will see that attention is a condition of thinking, and interest a condition of attention; that the mind is one and indivisible, and must be so treated if we would strengthen it. The mental as well as the physical nature is under law. When our teaching is in accord with this law, we shall find the forces of nature working for us; the child will become strong with the strength of nature.

Apprehension by the senses supplies, directly or indirectly, the material of all human knowledge; or, at least, the stimulus necessary to develop every inborn faculty of the mind. — *Helmholtz.*

The products of the senses, especially those of sight, hearing, and touch, form the basis of all the higher thought processes. Hence the importance of developing accurate sense concepts. . . . The purpose of objective

thinking is to enable the mind to think without the help of objects. — *Thomas M. Balliet.*

The understanding must begin by saturating itself with facts and realities. . . . Besides, we only understand that which is already within us. To understand is to possess the thing understood, first by sympathy and then by intelligence. Instead of first dismembering and dissecting the object to be conceived, we should begin by laying hold of it in its *ensemble*. The procedure is the same, whether we study a watch or a plant, a work of art or a character. — *Amiel.*

The action of the mind in the acquisition of knowledge of any sort is synthetic-analytic; that is, uniting and separating. These are the two sides, or aspects, of the one process. . . . There is no such thing as a synthetic activity that is not accompanied by the analytic; and there is no analytic activity that is not accompanied by the synthetic. Children cannot be *taught* to perform these knowing acts. It is the nature of the mind to so act when it acts at all. — *George P. Brown.*

Our children will attain to a far more fundamental insight into language, if we, when teaching them, connect the words more with the actual perception of the thing and the object. . . . Our language would then again become a true language of life, that is, born of life and producing life. — *Froebel.*

Voluntary attention is a habit, an imitation of natural attention, which is its starting-point and its basis. . . .

Attention creates nothing; and if the brain is barren, if the associations are meagre, it functions in vain. — *Ribot.*

How, indeed, can there be a response within to the impression from without when there is nothing within that is in relation of congenial vibration with that which is without? Inattention in such case is insusceptibility; and if this be complete, then to demand attention is very much like demanding of the eye that it should attend to sound-waves, and of the ear that it should attend to light-waves. — *Dr. Maudsley.*

Activity bears fruit in habit, and the kind of activity determines the quality of the habit. — *Alex E. Frye.*

If a teacher is full of his subject, and can induce enthusiasm in his pupils; if his facts are concrete and naturally connected, the amount of material that an average child can assimilate without injury is as astonishing as is the little that will fag him if it is a trifle above or below or remote from him, or taught dully or incoherently. — *G. Stanley Hall.*

Is it not evident, that if the child is at any epoch of his long period of helplessness inured into any habit or fixed form of activity belonging to a lower stage of development, the tendency will be to arrest growth at that standpoint and make it difficult or next to impossible to continue the growth of the child? — *William T. Harris.*

We must make practice in thinking, or, in other words, the strengthening of reasoning power, the constant object of all teaching from infancy to adult age, no matter what may be the subject of instruction. . . . Effective training of the reasoning powers cannot be secured simply by choosing this subject or that for study. The method of study and the aim in studying are the all-important things. — *Charles W. Eliot.*

Intellectual evolution is, under all its aspects, a progress in representativeness of thought. — *Herbert Spencer.*

Consciousness implies perpetual discrimination, or the recognition of likenesses and differences, and this is impossible unless impressions persist long enough to be compared with one another. . . . Impressions persist long enough to be compared together, and accordingly there is reason. — *John Fiske.*

Thinking is discerning relations; but we discern the relations of things. In order to discern relations we must compare; hence, our powers to think are our comparative powers. These are our faculties to discern relations. — *Dr. M' Cosh.*

Thought consists in the establishment of relations. There can be no relation established, and therefore no thought framed when one of the related terms is absent from consciousness. — *Herbert Spencer.*

Intelligence is virtually a correct classification. — *Dr. Maudsley.*

The thing is its relations, and although analytically we may separate them, attending now to this relation, now to that, we must never imagine the separation to be real. — *G. H. Lewes.*

All knowledge results from the establishment of relations between phenomena. — *J. B. Stallo.*

Every act of judgment is an attempt to reduce to unity two cognitions. — *Sir William Thomson.*

The primary element of all thought is a judgment which arises from a comparison. — *Francis Bowen.*

There is no enlargement of the mind unless there be a comparison of ideas one with another. — *Cardinal Newman.*

The extent or magnitude of a quantity is, therefore, purely relative, and hence we can form no idea of it except by the aid of comparison. — *Davies.*

Of absolute magnitude we can frame no conception. All magnitudes as known to us are thought of as equal to, greater than, or less than, certain other magnitudes. — *Herbert Spencer.*

Those who accept the above can hardly agree with the prevailing practices in the teaching of arithmetic.

It is hoped that the following brief presentation of mathematics as the science of relative magnitude will aid teachers in bringing mathematical teaching into accord with educational principles.

MATHEMATICS, — DEFINITE RELATIONS.

Mental advance from the vague to the definite. — Teaching which meets the needs of the developing mind must be successful. No other can be.

The marvelous progress of the child during the first five or six years of its life is largely due to free action and spontaneous attention; to the absence of demands unfavorable to growth.

We recognize the incapacity of the infant. We watch and minister to its growth by creating an environment fitted for calling forth its activities. So should we acquaint ourselves with the mental state of the child, as shown in his work, his play, his questions; in what he hears and sees; in what he does and in what he tries to do; in what he says and in what he does not say. From the basis of his experience and power our training should proceed.

Complex conceptions cannot be imposed upon a mind incapable of receiving them; neither can simple truths. Nothing is self-evident save to him who sees it. The child no more knows that things equal to the same thing are equal to each other, until he sees it to be so, than he knows that yellow and blue make green. He sees only that which he has the power to see.

The change from the helplessness of the babe to the power of the child of six is a constant miracle; but its powers are still relatively feeble.

Between the capacity for vague perceptions and for framing definite mathematical ideas there are many intervening stages.[1] The natural approach to each higher thought-product is through the lower one, which is its necessary antecedent.

The perception of equality is the basis of mathematical reasoning, — a condition of definite thinking. But a child sees things as longer or shorter, larger or smaller, before he is able to see their perfect equality or exact degree of inequality.[2] Until, without effort, he makes such discriminations as are expressed by the terms long, short, large, small, etc., he is not ready to make the discriminations expressed by twice, three times, $\frac{1}{2}$ or $\frac{1}{3}$.

Analysis dependent upon representative power. — Exact quantitative relations cannot be established without analyzing. Analysis fixes the at-

[1] "In early life the cerebral organization is incomplete. The period necessary for completion varies with the race and with the individual." — Prof. Tyndall.

[2] "The conception of exact likeness," remarks Mr. John Fiske, "is a highly abstract conception, which can only be framed after the comparison of numerous represented cases in which degree of likeness is the common trait that is thought about." — Cosmic Philosophy, vol. ii. p. 316.

tention in turn upon each part rather than upon the relation of the compared wholes. When the pupil enters upon the process of exact comparison he should be able to hold each term of the comparison so firmly that the necessary intrusion of a common measure will not efface either of them. Otherwise, the operations intended to throw into relief the precise relation of the magnitudes interpose as a cloud to render the relation invisible.

Place a measure in the hands of a pupil and set him to marking off spaces on this and that and counting them before he is ready for such work, before anything has been done to induce the habit of looking from one magnitude to another, and you absorb him in a mechanical process which turns the thought from the relational element with which mathematics deals. He may write, "The door is 8 feet high," when he has simply counted 8 spaces. But he has made no mathematical comparison, observed no relation, done little which tends to develop power to think. If we ask him to find exact relations before he has sufficient *representative power* to bring each term of the comparison into consciousness and approximate its relations unaided, the probability is that the *relation* of the magnitudes as wholes will not be seen at all.

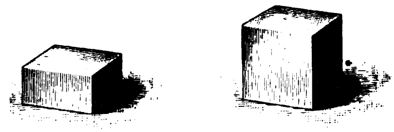
Premature attempts to initiate the pupil into the ideas of mathematics will bewilder him with

the mechanism of the subject and create a condition unfavorable to the perception of mathematical or any other truth.

"Not only is it true," says Herbert Spencer, "that in the course of civilization qualitative reasoning precedes quantitative reasoning; not only is it true that in the growth of the individual mind the progress must be through the qualitative to the quantitative, but it is also true that every act of quantitative reasoning is *qualitative in its initial stages.*"

Unity of subject. — The teacher must be clear as to what characterizes a science. Otherwise the essential may be lost sight of in the subordinate, and the energy of the pupil wasted in the effort to unite what should never have been separated.

A living apprehension of the fact that mathematics deals with *definite relations of magnitude* suggests the mode of beginning the study. It suggests the need of creating definite ideas; it forbids presenting things as isolated, independent,



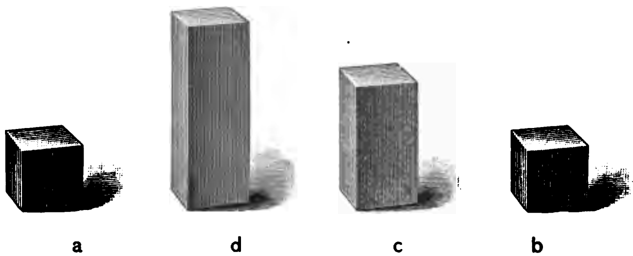
absolute in themselves. It does away with artificial distinctions between a fraction and an integer, by presenting each as a relation. Thus 2 is

the relation of a unit to another half as large; and one half is the relation of a unit to another twice as large.

A relation the result of a comparison. — To be conscious of a relation means more than to be conscious of the terms between which it exists. We may think of the taste of an orange or of a pear without connecting them in any way, but if we are considering their *relative* sweetness we must bring together in thought the taste of each; a comparison must take place before we can assert that one is sweeter than the other. So we may think of a certain line as one yard, of another as six inches, without ability to assert their *relative* magnitude. We may go further and note the fact that in one yard there are six six-inches, and still remain without any appreciation of their relative magnitude. Before we can assert this, the *intellectual* act which brings the shorter line before the mind as *equal* to one sixth of the longer must take place.

We cannot meet the demands of mathematics by observing things simply as distinct and separate. If relations are to come into consciousness, the comparing which brings them there must take place. An example may make this more clear. Suppose the magnitudes *a*, *b*, *c*, and *d*, to be before the child. He notes likenesses and differences in

them just as he does in colors, leaves, fruit, or anything to which he attends. Noting *d* and *a* he sees that *d* is greater than *a*, that *a* is less than *d*.



He has made comparisons and established relations, but not exact relations. These relations he expresses by the indefinite words greater and less. If, by measuring, he effects an exact comparison of *d* and *a*, he needs language for stating that the relation of *d* to *a* is 3; the relation of *a* to *d* is $\frac{1}{3}$. He may call *a* $\frac{1}{3}$ and *d* 1, or *d* 3 and *a* 1; or he may call *d* 12 and *a* 4, but their *relation remains unchanged.*

The thing is its relations. — Comparing *c* with *a* considered as 1, we call *c* 2. Comparing *c* with *d*, *c* becomes $\frac{2}{3}$, yet the magnitude *c* has not changed. The *a* which we dealt with as $\frac{1}{3}$ when thought of in relation to *d*, as $\frac{1}{2}$ in relation to *c*, we call 1 when compared with *b*, or with any other equal magnitude.

Just as the child learned to know a line as long in comparison with another, short in comparison with a third; to call a day warm or cold according to that with which it is compared, so he should learn to know a magnitude as 2 when seen in relation to a magnitude equal to its $\frac{1}{2}$; to see the same magnitude as 3 or 5 or $\frac{1}{5}$ when compared with other magnitudes.

Means of comparing. — Effecting an exact comparison requires analysis and synthesis, just as every act does which results in a judgment.

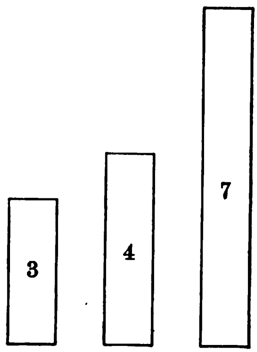
$\boxed{\quad 6 \quad} \quad \boxed{\ 4\ }$

In order to discover the relation of 4 to 6, we may separate the 4 and the 6 each into 2's. By the analysis (subtraction or division) we find 3 2's in the 6 and 2·2's in the 4. Since 2 is $\frac{1}{3}$ of 3·2's, we infer that 2 is $\frac{1}{3}$ of 6. (Why?) In order to make such an inference we must see that 3·2's equal 6, — synthesis (addition or multiplication).

From successive relations of equality we pass to the final act of relating, which brings 2·2's, or 4, before the mind as equal to $\frac{2}{3}$ of 6. The final thought is not of the 4 nor of the 6, nor of the relation of the measuring unit to either; but of the relation of the 4 to the 6. In no case have we established the relation sought until the com-

pared wholes are brought into consciousness *in that relation.*

Again, suppose we wish a child to discover the relation which exists between 7 and the sum of 3 and 4.

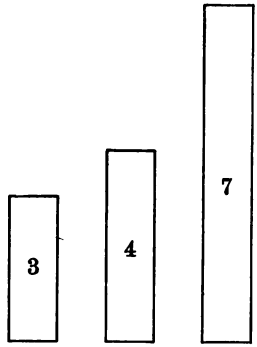


Mental acts must take place showing him in the 7 two magnitudes,—one equal to 4, the other equal to 3,—and bringing the 7 again before the mind as one quantity. But we must pass beyond these steps and bring the seven and the sum of 3 and 4 before the mind in the relation of equality. A judgment of relative magnitude must be formed which unites the compared terms.

Each of these judgments, like every other judgment, is the product of analysis and synthesis—of separating and uniting; of subtracting and adding; of dividing and multiplying. There is no real synthesis without analysis, no addition

without subtraction, no multiplication without division.

Conditions of comparing. — In comparing there must be ideas to compare. In presenting the magnitude 7, as well as 3 and 4, we are merely meeting that condition of thinking which requires that, in establishing a relation, each of the compared terms must be present in consciousness. Through this comparison the pupil learns to know 7 in one relation; through other comparisons he will enter into fuller knowledge of it.



Meaning of a word depends upon experience. — When the need of a name arises, give it. The principle is the same whether dealing with the qualitative or the quantitative. We do not leave the child without the name *water* because he does not know the elements of water. We tell him a certain object is a chair long before he has a complete idea of it. As his surroundings produce activity, he gradually comes to know special features of the chair, and to distinguish arm-chairs, rocking-chairs, etc.

The name alone can avail nothing;[1] but when

[1] Language attains definiteness for the individual only as it is associated with definite ideas. The square is a definite figure;

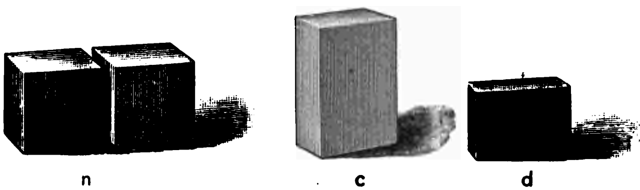
it will be serviceable in focusing the attention, in aiding the child to retain his grasp of a thing, and thus in facilitating his investigations, it should be freely given.

Thought and expression are inseparable. Words without ideas are dead; images without words are elusive. The most effective method of mastering the means of expression in mathematics, or in any other subject, is the exercise of the mind upon realities, — in mathematics, the realities are the relations of magnitude.

Using divided magnitudes obscures wholes, weakens sense of coexistence. — By presenting divided magnitudes (see *n* below), we destroy the wholes we wish compared, and call upon the child for a synthesis for which he is not prepared. The problem does not require him to make a comparison of the *magnitudes*, but merely to count the *how many*. We force upon the attention isolated units and operations for which the mind has no need, and which, by being thus pushed into the

but the child may handle many squares and repeat the definition of a square many times without any *feeling* of its definiteness. If we taught the child to say that the sum of 3 and 4 = 7, without his mentally seeing it to be so, we should be presenting symbols without significance. To refuse to give the name 7 to the magnitude in this particular relation, because the learner is not fully conscious of the meaning of the term, is as if we refused to allow a child to talk of a star because his idea of it is not that of the astronomer.

foreground, tend only to intellectual chaos. A synthesis not accompanied by analysis must be artificial. There can be no real synthesis without analysis.



In observing *n* (the divided magnitude) does the child consider the *relative magnitude* of the units or the *how many?* In comparing *c* (the undivided magnitude) with *d*, what receives the primary attention, the *how many* or the *relative size?*

The child grasps a dollar or a dozen as a unit, untroubled by its composition. So it should grasp a 12, a 17, a 100, a $\frac{1}{3}$, or a 7. So it will if you bring them into consciousness as wholes.[1] If you wish a pupil to note the relation between the length and the width of a desk; or between a 12 and a 3; a 1,200 and a 300; a 68 and a 17; or

[1] "Now, the fact is, that all objects of apprehension, including all data of sense, are *in themselves*, *i.e.* within the act of apprehension, essentially continuous. They become discrete only by being subjected, arbitrarily or necessarily, to several acts of apprehension, and by thus being severed into parts, or coördinated with other objects similarly apprehended into wholes."— J. B. Stallo.

a $\frac{1}{2}$ and a $\frac{1}{8}$, what are, in each case, the wholes to which you wish him to attend?

If, instead of bringing the terms of the comparison before the mind as related wholes, we require the learner to begin by constructing them from the parts,[1] we destroy for him the continuity of the magnitudes. Consciousness is occupied with a succession of separate units, and but a vague sense of the relations of the given magnitudes is awakened.

Undivided magnitudes; use induces analysis and synthesis. — It must not be supposed that the mere use of undivided magnitudes will insure the perception of mathematical relations;[2] but it fosters such perception. It is a condition of presentation in accord with the familiar fact that the

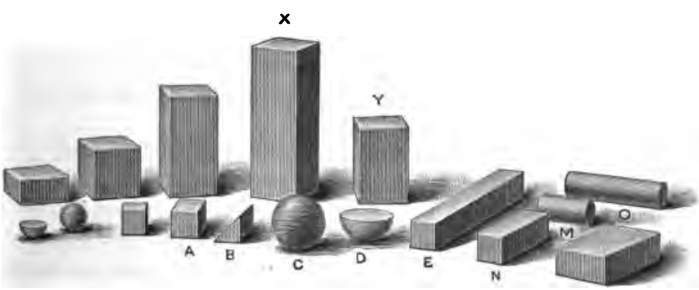
[1] "Where the parts of an object have already been discerned, and each made the subject of a special discriminative act, we can with difficulty feel the object again in its pristine unity; and so prominent may our consciousness of its composition be, that we may hardly believe that it ever could have appeared undivided. But this is an erroneous view, the undeniable fact being that *any number of impressions, from any number of sensory sources, falling simultaneously on a mind* WHICH HAS NOT YET EXPERIENCED THEM SEPARATELY, *will fuse into a single undivided object for that mind.*" — Wm. James.

[2] The material provided for mental nutrition is most important. But there is danger of relying too exclusively upon special methods and intrinsic values. Undue reliance upon any means or subject may blind us to the fact that the educational process is not going on at all.

mind moves from the whole to the part and back again to the whole; that it analyzes through a desire for more intimate knowledge, in order that it may reach a better synthesis. We should present as wholes the magnitudes whose relations we wish established, and leave the way open for those successive acts of analysis and synthesis by which such relations are established.

Freeing the mind from the concrete. — Noting the same relation between many different magnitudes tends to free the mind from the concrete and the particular, and to make the relations the objects of thought.

Thus the pupil sees magnitudes differing greatly



in size, but discovers that 2 is the relation not only of c to d, but of x to y, of a to b, of o to m, and of e to n; he notes the unlikeness of the separate pairs, the likeness of their relations; he

is asked for inference after inference which turns attention to the ratio of the units.[1] Gradually he learns to know magnitudes in the only way that they can be known, — *in relation.* The simple ratios of mathematics become *real* to him.

Giving varying names to the units, as a 12 and a 6, a $\frac{1}{3}$ and a $\frac{1}{6}$, a 100 and a 50, aids in separating accidental from essential relations, and in preventing the error of mistaking the relative for the absolute.

Through many, very many experiences, fitted for developing the power, he becomes able to dissociate the relation from the thing, and to deal with the 2, the 3, the $\frac{1}{2}$, the $\frac{1}{3}$, etc., as uniform relations upon which far-reaching inferences may be based.[2]

[1] "The higher processes of mind in mathematics lie at the very foundation of the subject." — Sylvester.

[2] "The peculiarity of abstract conceptions is that the matter of thought is no longer any one object, or any one action, but a trait common to many; and it is, therefore, only when a number of distinct objects or relations possessing some common trait can be represented in consciousness, that there becomes possible that comparison which results in the abstraction of the common trait as the object of thought." — John Fiske.

"The development of ideas is the slow, gradual result of continuous judgment." — Francis Bowen.

"What is associated now with one thing and now with another tends to become dissociated from either, and to grow into an object of abstract contemplation by the mind." — Wm. James.

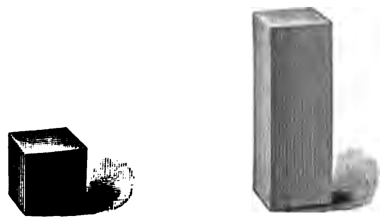
Inference must succeed perception.—The importance of bringing simple basic ratios definitely into consciousness is better understood when we look beyond them.[1]

The development of mathematics within the mind, and the development of the mind by means of mathematics, are alike impossible without that thinking, relating, reasoning, by which the mind "produces from what it receives." From the beginning we must address the *mind* and not one function; give opportunity for inference to succeed perception. By unduly crowding the sensing and recording of ratios we may so handicap the mind that it cannot move. As law or principle serves the man of science, so each simple truth should serve the child in lighting the way to other truths.

By means of perceived relations we must pass to the inferring of relations. For example, it is not enough for the child to see the relation of *d* to *a* and of *a* to *d*. From these perceptions he must infer other relations. Rightly taught, such inferences as that the weight of *d* equals 3 times the

[1] "And if we neglect to educe the fundamental conception on which all his ulterior knowledge must depend we not only sow the seed of endless obscurity and perplexity during all his future advance in this science, but we also weaken his reasoning habits . . . and thus make our mathematical discipline produce, not a wholesome and invigorating but a deleterious and perverting effect upon the mind."—Whewell.

weight of a; that 3 times as many inch cubes can be cut from d as from a; that d will yield 3 times as much ashes as a; that the cost of d equals 3 times the cost of a; that the weight of a equals $\frac{1}{3}$ the weight of d; that a will yield $\frac{1}{3}$ the amount of



a d

ashes that d will yield; that the cost of a equals $\frac{1}{3}$ the cost of d, will follow naturally and readily upon the perception of the ratio of d to a and of a to d. They will never follow without the generating conditions, and the generating conditions are perceptions of exact relations. Upon these equations, made known by the activity of the mind upon the magnitudes themselves, all mathematical deduction depends.

We frequently hear it said, "Is it not a proof that the child sees the conditions when he says that d will cost $3\,x$ if a cost x?" Or, if this is not enough, he can tell you that, "Because d is 3 times a, etc." Experience shows, however, that many pupils who can do all this will tell you a little later that it will take 3 boys 3 times as long to do a piece of work as it will one boy;

that the weight of a 2-inch cube is twice that of an inch cube. As they advance, their *seeming* inaptitude for mathematics becomes more marked. Why is this? Because a wrong direction was given to the mind in the beginning; because mechanical processes for securing results were substituted for those experiences which create ideas of equality and of exact ratio; because using objects merely to teach children to count and to manipulate numbers,[1] instead of presenting them in such a way as to attract attention to their *relative magnitude,* leaves the mind without any basis for the deductions which are demanded.

Out of number nothing comes save number. If we ask for conclusions concerning quantity we must see that the mind possesses a basis[2] for those conclusions. The indispensable groundwork of reasoning is the definite mental representation of the relation upon which an inference rests; and mathematical inferences rest upon ratios.

Clear imaging; clear thinking; correct conclusion. —The material upon which the mind can act from time to time depends upon its growing

[1] "It would indicate a radically false idea of number to wish to employ it in establishing the elementary foundations of any science whatever; for on what would the reasoning in such an operation repose?" — Comte.

[2] "The attempt to found the science of quantity upon the science of number I believe to be radically wrong and educationally mischievous." — Wm. K. Clifford.

power to represent in thought the conditions upon which conclusions follow.

Pupils accept the statement that it will take twice as long to paint a 2-inch square as a 1-inch square because they do not represent the squares mentally.

If the pupil has been trained so that it is his *habit* to make the necessary mental representations he will see for himself that if x is the number of yards of carpet $1\frac{1}{2}$ yards wide required for a floor, $2\,x$ yards $\frac{3}{4}$ of a yard wide will be needed. No wordy explanation will be required. Yet pupils fail constantly in such simple exercises. They cannot make comparisons, because they have in their minds no images of the things they are to compare. They cannot deduce from symbols the relations of reals.[1] Asking pupils to reason about things which they do not see mentally, is asking the impossible and can only lead to confusion and discouragement. The power of representative thought, of imaging, underlies all intellectual progress, and we cannot prepare the mind for abstract thought without developing this power.

Mathematics deals with realities. — However divergent may be the lines of mathematical

[1] "How accurate soever the logical process may be, if our first principles be rashly assumed, or if our terms be indefinite or ambiguous, there is no absurdity so great that we may not be brought to adopt it." — Dugald Stewart.

thought, their beginnings are sensible intuitions, — that is, the ideas of magnitude must be based on perceptions; and however long the line, its extension is in all cases by means of successive acts of comparing and inferring.

Sylvester finds that "The study of mathematics is unceasingly calling forth the faculties of observation and comparison; that it has frequent recourse to experimental trial and verification; and that it affords a boundless scope for the highest efforts of imagination. . . . I might go on," he says, "piling instance upon instance to show the paramount importance of the faculty of observation to the process of mathematical discovery."

"Mathematics," says Mr. Lewes, "is a science of observation, dealing with reals, precisely as all other sciences deal with reals. It would be easy to show that its method is the same." The reals are the relations of magnitude.

The order of truth changes; the mental action which embraces it remains the same. We note the likenesses of two leaves or the exact likenesses of two magnitudes; in each case we have a basis for inference obtained by comparing. When we turn to exact likenesses, we enter the domain of mathematics.

Objects unfitted to awaken mathematical ideas. — Were we concerned simply with the *number* of

things, beans, shoe-pegs, shells, leaves, pebbles, chairs, or the legs of frogs might serve as well as anything. But mere numerical equality will not serve as a basis for mathematical reasoning; exact results cannot be founded upon it.

Dealing with units, without regard to their equality or inequality; considering them only as distinct things; and reaching results true only numerically, has been called the *indefinite calculus;* but the indefinite calculus furnishes no basis for mathematical reasoning. If arithmetic is made merely a means of teaching number, and operations with number, it should receive but brief time in the common-school course. Very little of it will suffice for the ordinary vocations of life. The cases in which mere numerical relations are considered are so simple as scarcely to stir the mind.[1]

A superficial knowledge of mathematics may lead to the belief that this subject can be taught incidentally, and that exercises akin to counting the petals of a flower or the legs of a grasshopper are mathematical. Such work ignores the fundamental idea out of which quantitative reasoning

[1] In regard to the how many, to work which does not deal with definite relations, Comte said, "This will never be more than a point, so to speak, in comparison with the establishment of relations of magnitude of which mathematical science essentially consists. . . . In this point of view, arithmetic would disappear as a distinct section in the whole body of mathematics."

grows — the equality of magnitudes.[1] It leaves the pupil unaware of that relativity which is the essence of mathematical science. Numerical statements are frequently required in the study of natural history, but to repeat these as a drill upon numbers will scarcely lend charm to these studies, and certainly will not result in mathematical knowledge.

Vague ideas of the unlikeness of a rhomboid, a square, and a trapezium may be gained by counting them, and so may vague ideas of the relations of magnitude. If definite ideas of color, form, or weight come from counting and learning tables, then definite ideas of quantitative relations may come in the same way.

Turning from the numbering of things to their mathematical comparison, we see at once why plants and animals are not well adapted for our purpose. In them, that which is material is obscured by that which is irrelevant.[2] It is difficult for the undeveloped mind to view these objects

[1] "Equations constitute the true starting point of arithmetic." — Comte.

"The fundamental ideas underlying all mathematics is that of equality." — Herbert Spencer.

[2] "The visible figures by which principles are illustrated should, so far as possible, have no accessories. They should be magnitudes pure and simple, so that the thought of the pupil may not be distracted, and that he may know what feature of the thing represented he is to pay attention to." — Committee of Ten."

in their mathematical aspect. Their differences in magnitude are not easily appreciated by the senses. Their exact measurement is not easy. They lend themselves to accurate imaging far less readily than simple magnitudes, and do not result in those mental states which would be created were mathematical relations brought conspicuously and impressively into the pupils' experiences.

That mathematics enters into other sciences is understood. The fruitfulness of physics for the teacher of mathematics is apparent. Advancing science is constantly making more clear the interdependencies of the various sciences. Each aids in the development of the others.[1] But it does

[1] "Although each science throws its light on every other, owing to the interdependence of phenomena and the community of consciousness, yet . . . phenomena are *in*dependent not less than *inter*dependent. Mathematics cannot receive laws from chemistry, nor physics from biology; the phenomena studied in each are special." — Lewes.

"This unification of all the modes of existence by no means obliterates the distinction of modes, nor the necessity of understanding the special characters of each. . . . If we recognize the *one* in the *many*, we do not thereby refuse to admit the *many* in the *one*." — Lewes.

"Sciences are the result of mental abstraction, being the logical record of this or that aspect of the whole subject-matter of knowledge. As they all belong to one and the same circle of objects, they are one and all connected together; as they are but aspects of things, they are severally incomplete in their relation to the things themselves, though complete in their own idea and for their own respective purposes; on both accounts they at once need and subserve one another." — Cardinal Newman.

not follow that different classes of ideas will be equally excited by the same objects.

The result of trying to call forth mathematical ideas by means of phenomena whose exact measurement is beyond the power of the pupil, is very similar to the result when no pretence is made of founding deduction upon perception. Why should it not be? In neither case do mathematical relations come definitely into consciousness.

What objects will excite definite ideas? — Things whose *exact* relations can be most readily seen; things which can be most accurately imaged and exactly compared; things which tend most to excite definite intuitions and to result in definiteness of mind, should be given precedence in elementary instruction in mathematics.

Comte observes, "The only comparisons capable of being made directly, and which could not be reduced to any others more easy to effect, are the simple comparisons of right lines." This is apparent to whoever gives thought to the matter.[1]

[1] "On tracing them back to their origins, we find that the units of time, force, value, velocity, etc., which figures may indiscriminately represent, were at first measured by equal units of space. The equality of time becomes known either by means of the equal spaces traversed by an index, or the descent of equal quantities (space-fulls) of sand or water. Equal units of weight were obtained through the aid of a lever having equal arms (scales).

Since the measurement of all magnitudes is reducible to measurements of linear extension, and since the comparison of linear units alone reveals that perfect equality upon which the science of mathematics is built, — since by such comparisons and only by them do we obtain the original materials of mathematical thought, since these experiences alone give rise to those abstract conceptions which enable us to use numbers intelligently,—it follows that definite magnitudes should furnish the objective stimulus in laying a basis for mathematical knowledge. Out of ratios established by comparing right lines the ratios of surfaces and solids are inferred, and also the quantitative relations of units of value, force, and, in short, of all other magnitudes.

The problems of statics and dynamics are primarily soluble, only by putting lengths of lines to represent amounts of forces. Mercantile values are expressed in units which were at first, and indeed are still, definite weights of metal; and are, therefore, in common with units of weight, referable to units of linear extension. Temperature is measured by the equal lengths marked alongside a mercurial column. Thus, abstract as they have now become, the units of calculation, applied to whatever species of magnitude, do really stand for equal units of linear extension, and the idea of coextension underlies every process of mathematical analysis. Similarly with coexistence. Numerical symbols are purely representative; and hence may be regarded as having nothing but a fictitious existence." — Spencer, Principles of Psychology, vol. ii. p. 38.

"Whenever I went far enough I touched a geometrical bottom." — Prof. Sylvester, Address British Association, 1869.

Means of passing beyond the range of perception. — It is the definite relations of magnitudes established by means of solids, surfaces, and lines, that enable us to conceive or interpret the relations of quantities which cannot be brought within the range of perception. The ratios which we actually see are few, but out of these grows the science of mathematics.

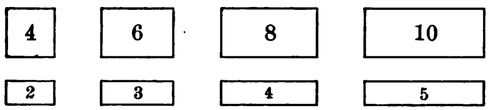
These primary relations, then, should be so repeatedly felt, so ingrained, that they will become elements in the mental life. This is possible only by confronting the pupil again and again with the conditions which force upon him the methods and ideas of mathematics. He should become so identified with the kind of relations dealt with, that the abstract terms in which he afterwards reasons will be *truly representative.* Otherwise, he will restrict and misapply them. It is the *certainty* of the *seen* that makes us rationally certain of the unseen.

The basis of drills the perception of relations. — It is well understood that the use of language must become automatic if the mind is to move freely in the discovery of laws and principles.

How is this needful familiarity with the means of making quantitative comparisons to be provided for? Not, certainly, by treating the means as though it were the end; not by forcing premature

drill upon tables and routine work in combining and separating symbols. This is to ignore mathematics, to ignore natural sequences, both within and without the mind. Its tendency is to prevent energy from rising to that higher kind of power of which an intelligent being is capable.

The drills should harmonize with the dominant idea of the subject and meet the conditions which favor retention without interfering with growth.



In his observing and comparing, the pupil has dealt with the ratios 2, 3, 4, etc. He has seen that the ratio of 4 to 2 equals the ratio of 6 to 3, of 8 to 4, of 10 to 5, of $\frac{1}{2}$ to $\frac{1}{4}$, etc. We bring these equal ratios together in the same table and associate them in his mind. Making the *common* thing, the *ratio*, prominent, unifies the work and relieves the memory. Grouping like ratios in the drills is analogous to the grouping required in solving problems. Thus, the pupil sees that the relation of the cost of 6 acres to the cost of 2 acres is equal to the relation of their areas. From one truth he passes to another, and brings the differing ideas into unity. The drills should emphasize this sense of likeness in the midst of

difference without interfering with the flexibility of the mind.

Drill work should be a means of increasing mental power by training the eye to quickness and accuracy, and the mind to attend closely and image vividly.

In every exercise the first thing to secure is a clear mental picture. When the pictures are distinct, work for rapidity. What is to be recognized at sight should be taken in through the eye.[1] The visual image will be dimmed and blurred, and

[1] "A common error, into which beginners are apt to fall, is to try to combine, and therefore to confuse, the two methods of remembering, by sight and by sound." — Dr. M. Granville.

"When a child first sees a thing, it takes it in by the eye; when it first hears a thing, it takes it in by the ear; in each case the whole mind is concentrated on the sensation, which, as Dr. Carpenter says, 'is the natural state of the infant.' But as soon as education begins, all this is changed, and the mind, instead of being concentrated upon one thing, is distracted by several."— Kay.

"We must attend to the formation of the original impression . . . and recall it in its entirety afterwards." — Kay.

"Nothing needs more to be insisted on than that vivid and complete impressions are all-essential." — Herbert Spencer.

"There can be no doubt as to the utility of the visualising faculty when it is duly subordinated to the higher intellectual operation. A visual image is the most perfect form of mental representation wherever the shape, position, and relations of objects in space are concerned."— F. Galton.

"The more completely the mental energy can be brought into one focus, and all distracting objects excluded, the more powerful will be the volitional effort." — Dr. Carpenter.

hence imperfectly remembered, if we attempt to call the ear into action at the same time that we address the eye.

The way not to succeed in memorizing the tables is to repeat so many different impressions in the same exercise that none of them are distinct; to confuse eye and ear training; to make the work so difficult that it cannot be done easily and

"It is a matter of common remark that the permanence of the impression which anything leaves on the memory is proportioned to the degree of attention which was originally given it." — D. Stewart.

"Most persons find that the first image they have acquired of any scene is apt to hold its place tenaciously." — F. Galton.

"The habit of hasty and inexact observation is the foundation of the habit of remembering wrongly." — Dr. Maudsley.

"No ideas can long be retained in the memory which are not deeply fixed by repetition." — Joseph Payne.

"The leading principle is to learn very little at a time, not in a loose, careless way, but perfectly." — F. Prendergast.

"A few such items must be memorized and reviewed daily, adding a small increment to the list as soon as it has become perfectly mastered." — W. T. Harris.

"We usually attempt to master too much at once, and hence the impressions formed in the mind lack clearness and distinctness." — Kay.

"All improvement in the art of teaching depends on the attention that we give to the various circumstances that facilitate acquirement or lessen the number of repetitions for a given effect." — Prof. Bain.

"It is not enough that impressions be received; they must be fixed, organically registered, conserved; they must produce permanent modifications in the brain. . . . This result can depend only on nutrition." — Th. Ribot.

quickly; to drill once or twice a month; and to prolong the exercise until the power of attention is exhausted.

The way to succeed is to develop vivid mental pictures, and to fix these pictures by bringing them again and again before the mind.

Briefly summarized, we may say: Reasoning in arithmetic establishes equality of relations; reasoning in any subject, equality or likeness of relations.

We know magnitudes only in relation; and the purpose of mathematical science is to establish *definite relations* between magnitudes. The fundamental operation is comparison. Out of the relations established by comparison grow inferences.

Only through the *activity* of the mind in observing and comparing can those equations be formed which are the groundwork of reasoning, the basis of advance from relations seen to relations which lie beyond the range of perception.[1]

That quantity is a ratio between terms which are themselves relative; that mathematics is not

[1] "The domain of the senses, in nature, is almost infinitely small in comparison with the vast region accessible to thought which lies beyond them. . . . By means of data furnished in the narrow world of the senses, we make ourselves at home in other and wider worlds, which are traversed by the intellect alone. . . . We never could have measured the waves of light, nor even imagined them to exist, had we not previously exercised ourselves among the waves of sound." — Prof. Tyndall.

concerned with things as separate and absolute; that it deals only with relations, are truths which have often been pointed out, but which the work of the school shows to be felt by few.

In the light of these ideas, those arbitrary divisions, so fatal to the continuous unfolding of thought, are seen to belong to our language and our schemes of study, rather than to the subject.

Make definite relations the basis, and the integer and the fraction are each seen as a ratio; geometry, arithmetic, and algebra merge insensibly into one another. With definite relations as the center, it becomes clear that if we would teach *mathematics*, and not the mere mechanism of the subject, we must look to the development of the representative and comparative powers. Only thus can we lift arithmetic from a matter of memory, routine, and formula to its rightful place as a means of enlarging the mind.

PRIMARY ARITHMETIC.

FIRST STEPS.—SENSE TRAINING.

Finding solids.—Place spheres, cubes, cylinders, and other forms of various sizes in different parts of the room where the children can find them.

Show a sphere to the pupils. Ask:

1. What is this?

Find other balls or spheres.

Find a larger sphere than this. Find smaller ones.

2. Name objects like a sphere. Example: An orange is like a sphere.

3. What is the largest sphere that you have seen?

What is one of the smallest spheres that you have seen?

4. To-morrow tell me the names of spheres that you see when going from school and at home.

Ask, to-morrow, for the names of the objects and where they were seen.

5. What is the largest sphere you found? What is the smallest?

Review and work in a similar way with other solids.

"He should at first gain familiarity through the senses with simple geometrical figures and forms, plane and solid; should handle, draw, measure, and model them; and should gradually learn some of their simpler properties and relations." — Committee of Ten.

Children recognize objects similar in form, color, etc., before they desire or have the ability to express what they see.

Until a child can readily select a form he is not ready to make a statement of what he has found. Let the approach to telling be through doing; through the activity of the pupil in discriminating and relating.

The teacher, and such pupils as are able, should use the proper terms, so that pupils who have not heard the terms may learn to apply them. Children can discover likenesses and differences — relations — but not the terms in which they are expressed. They should learn the terms unconsciously by living in an atmosphere where they are used. Since we think most easily in the names we have first and most familiarly associated with a thing, the right

term should be used from the beginning. Providing fitly for expressing is an important means of arousing self-activity.

The different exercises are to be continued from day to day, as the growing interest and powers of the child suggest, and until there is skill in performing and ease in expressing. The teacher should know the condition of the pupil's mind. His expression is the index to his mental state. Avoid anything which will tend to substitute mechanical expression for real expression. Any form which is not the outgrowth of what is within, which is not the genuine product of free activity, will mislead the teacher and weaken the child.

"Forms which *grow* round a substance will be true, good ; forms which are consciously *put* round a substance, bad. I invite you to reflect on this." — Carlyle.

Finding colors. — Tests in color should be given before the more formal work suggested below. For example : Group cards of the same color and threads of worsted.[1]

Provide ribbons, worsted, cards, etc., of different colors, to be found by pupils when looking for a particular color.

Pin or paste squares of standard red and orange where they can be seen. Pin the red above the orange.

1. Find things in the room of the same color as the red square. What things can you recall that are red ?

[1] These exercises are not to teach color, but are to train pupils to visualize, to attend, to compare, and to secure greater freedom in expressing through noting different relations. All pupils need such work before beginning the usual studies of the primary school. They lack needful elementary ideas, which must be obtained through the senses. The range of the perceptions needs to be widened.

2. Look at the orange square. Find the same color elsewhere in the room. Recall objects that have this color.

3. Close the eyes, and picture or image the red square. Now the orange square.

4. Which square is above? Which below? Name the two colors.

5. To-morrow bring something that is red and something that is orange. Also tell the names of orange or red objects that you see in going to and from school.

Pin or paste a square of yellow below the orange.

1. Look at the yellow. Find the same color in the room. Recall objects having this color.

2. Look at the red, then the orange, then the yellow. Close the eyes and picture the colors one after another in the same order.

Cover the squares.

3. Which color is at the top? At the bottom? In the middle?

4. Name the three, beginning at the top. Name from the bottom.

5. Which color is third from the top? Second from the top? Third from the bottom?

6. To-morrow bring something that is yellow and tell me the names of things that you have seen that are yellow.

Add a square of green.

1. Find green. Recall objects that are green.

2. Try to see the green square with the eyes closed.

3. Look at the four colors.

4. Think of the four, one after another, with the eyes closed.

Cover the squares.

5. Think the colors slowly from the top down. From the bottom up.

6. Name the colors from the top down. From the bottom up. Which is second from the top? Third from the bottom? Second from the bottom?

7. Which color do you like best?

Add a square of blue and work in the same manner with the five as with the four.

Add a square of purple.

Work for a few minutes each day until the colors can easily be seen mentally in the order given.

Show a standard color. Have pupils find tints and shades of this color, and tell whether they are lighter or darker than the standard.

Have pupils bring things that are shades or tints of standard colors.

Using colored crayon or water-colors, have pupils combine primary colors and tell whether the result is darker or lighter than the standard secondary color. Example: Mix red and yellow. Is the result darker or lighter than the standard orange?

Why is it one of the first duties of the schools to test the senses and to devise means for their development?

Handling solids. — Cover the eyes.

Have a pupil handle a solid. Take it away.

Uncover the eyes. Pupil finds a solid like the one handled.

Cover the eyes.

Give a pupil a solid. Take it away. Give him another.

Are the solids alike?

Which is the larger? Which is the heavier?

Repeat the exercise from day to day.

Judgment and memory should be carefully cultivated through the sense of touch as well as through the sense of sight. Touch and motion give ideas of form, distance, direction, and situation of bodies. "All handicrafts, and after them the higher processes of production, have grown out of that manual dexterity in which the elaboration of the motor faculty terminates."

Similar solids. — Have a pupil select a solid and think of some object like it. Have other pupils guess the name of the object.

Ex.: I am thinking of something like a sphere.

Is it an orange?

No, it is not an orange.

Is it a ball of yarn?

It is not.

Relative magnitudes. — Place a number of solids on the table.

1. Find the largest solid. Find the smallest solid.

2. Find solids that are larger than other solids.

Ex.: This solid is larger than that one.

Find solids that are smaller.

3. Name objects in the room larger than other objects.

Ex.: That eraser is larger than this piece of chalk.

Name objects less than other objects.

4. Give names of objects at home that are smaller than other objects.

Ex.: A cup is smaller than a bowl.

5. Recall objects that are larger than other objects.

Ex.: An orange is larger than a peach. Some beetles are larger than bees.

6. What animals are larger than other animals?

7. Recall objects that are smaller than other objects.

Ex.: A base ball is smaller than a croquet ball.

8. Find the largest pupil in the class. The smallest.

9. To-morrow tell me the names of objects that are larger than other objects and the names of others that are smaller.

1. Find things that are higher than other things in the room.

Ex.: The door is higher than that table.

2. Find the tallest pupil. The shortest.

Compare heights of pupils.

Ex.: Mary is taller than Harry.

Compare the heights of other objects.

3. Recall objects that are longer than other objects.

4. What leaves are longer than they are wide? What leaves are wider than they are long?

5. To-morrow tell me the names of other leaves that are longer than they are wide.

Cutting.—Let the pupils at first cut and draw what they choose. After a number of daily exercises, when they have gained some command of the muscles, let them try to cut in outline objects which you place before them or which they have seen. Let the work be simple.

The drawing and cutting should be done freely, without the restraint of definiteness. If you ask more than the pupil can easily represent, the strained, unnatural tension interferes with free muscular action. In the slow and painful effort to represent perfectly, the mind is absorbed in the parts and is prevented from seeing the whole. A *premature* demand for definite action is a fundamental error, in that it separates thought from expression.

"The imperative demand for finish is ruinous because it refuses better things than finish."—Ruskin.

"Of course one cannot understand a child's picture-speech at once, any more than one can his other utterances. We must study and learn it."—H. Courthope Bowen.

Building. — Have pupils build prisms equal to other prisms.

Teacher shows a prism and the pupils build.

Hold the attention to the relative size. This is the mathematical element.

Avoid the analysis of solids until the habit of recognizing them as wholes is formed. Do not ask for number of surfaces, lines, corners, etc. Such questions, if introduced *prematurely*, tend to destroy self-activity, to interfere with judgments of relative size and with the power to see relations.

"Analysis is dangerous if it overrules the synthetic faculty. Decomposition becomes deadly when it surpasses in strength the combining and constructive energies of life, and the *separate* action of the powers of the soul tends to mere disintegration and destruction as soon as it becomes impossible to bring them to bear as *one* undivided force." — Amiel.

Ear training. — Have pupils listen and tell what they hear.

Have pupils note sounds when various objects are struck.

Pupils close eyes. Teacher strike one of the objects. Pupils tell which was struck.

Teacher strike two or more objects.

Pupils tell by the sound the order in which they were struck.

Train pupils to recognize one another by their voices and by the sounds made in walking.

Pupils close eyes and listen.

Drop a ball or marble two feet, then three.

Pupils tell which time it fell the farther.

"There are two ways, and can be only two, of seeking and finding truth. . . . These two ways both begin from

sense and particulars; but their discrepancy is immense. The one merely skims over experience and particulars in a cursory transit; the other deals with them in a due and orderly manner." — Bacon.

"It appears to me that by far the most extraordinary parts of Bacon's works are those in which, with extreme earnestness, he insists upon a *graduated* and *successive* induction as opposed to a hasty transit from special facts to the highest generalizations." — Whewell.

Touch and sight training. — Pupils handle solids:

1. Find one of the largest surfaces of each solid.

Ex.: This is one of the largest surfaces of this solid.

2. Find one of the smallest surfaces.

3. Find surfaces that are larger than other surfaces.

Ex.: This surface is larger than that one.

4. Find surfaces that are smaller than other surfaces.

5. Compare the size of other surfaces in the room.

6. Find the largest surface or one of the largest surfaces in the room.

7. Close the eyes, handle solids, and find largest and smallest surfaces.

8. Cover the eyes; handle and tell names of blocks and of other objects.

The exercises for mental training are only suggestive of many others which teachers should devise. Be sure that the exercises are suited to the learner's mind, and to his physical condition.

Visualizing. — Place on the table three objects, for example: A box, a book, and an ink-bottle.

1. What can you tell about the box? About the book? About the ink-bottle? Which is the heaviest? Which is the lightest? Which is the largest?

2. Look at the three objects carefully, one after another.

3. Close your eyes and picture one after another.

Cover the objects.

4. Think the objects from right to left. From left to right.

5. Name the objects from right to left. From left to right.

6. Which is the third from the right? The second from the left?

"Our bookish and wordy education tends to repress this valuable gift of nature, — visualizing. A faculty that is of importance in all technical and artistic occupations, that gives accuracy to our perceptions and justness to our generalizations, is starved by lazy disuse, instead of being cultivated judiciously in such a way as will, on the whole, bring the best return. I believe that a serious study of the best method of developing and utilising this faculty without prejudice to the practice of abstract thought in symbols is one of the many pressing desiderata in the yet unformed science of education." — Francis Galton.

When the position of every object in the group can easily be given from memory, place another object at the left or right. Add not more than one object in an exercise unless the work is very easy for the pupils.

When a row of five is pictured and readily named in any order, begin with another group of five. Each day review the groups learned, so as to keep them vividly in the mind.

Questions or directions similar to the following will test whether the groups are distinctly seen:

Picture each group from the right. Name objects in each from the right.

In the third group, what is the second object from the left?

What is the middle object in each group? What is the largest object in each group?

When four or five groups can be distinctly imaged, this exercise might give place to some other.

Finding circles. — Show pupils the base of a cup, a cylinder, or a cone, and tell them that it is a circle.

Conduct the exercises so that the doing will call forth variety of expression in telling what is done.

The correct use of the pronouns, verbs, etc., will thus be secured without waste of the pupils' energy. What the pupils see and do should lead to statements similar to the following:

That circle is larger than this one. I have found a circle that is larger than that one. Helen has found a circle larger than that one. He has found a circle smaller than this one. They have found circles larger than this one.

1. Find circles.
2. Find circles that are larger than others. Find circles that are smaller.
3. Find the largest circle in the room.
4. Find one of the smallest.
5. Find circles in going to and from school and at home, and tell me to-morrow where you saw them.

Finding forms of the same general shape as those taken as types is of the highest importance. Unless this is done pupils are not learning to pass from the particular to the general. They are not taught to see many things through the one, and the impression they gain is that the particular forms observed are the only forms of this kind. Unless that which the pupil observes aids him in interpreting something else, it is of no value to him. Teaching is leading pupils to discover the unity of things.

Finding rectangles. — Show pupils rectangles (faces of solids), and tell them that such faces are *rectangles*.

1. Find other rectangles in the room.

Ex.: This blackboard is a rectangle.

2. Find larger and smaller rectangles than this one.

3. Find square rectangles. Find oblong rectangles.

Finding triangles. — Show the pupils the base of a triangular prism or pyramid.

The base of this solid is a triangle.

1. Find triangles in the room.

2. Find triangles that are larger and smaller than other triangles.

Finding edges or lines. — Place solids where they can be handled.

1. Show edges of different solids.

Show one of the longest[1] edges of the largest solid.

[1] The form of the solid will, of course, determine the adjective to use. Every lesson should help to familiarize the child with correct forms of speech.

2. Look for the longest edges in each of the solids.

3. Show the longer edges of other objects in the room.

Ex.: This and that are the longer edges of the blackboard.

4. Show the shorter edges of different objects.

5. Find edges of different solids and tell whether they are longer or shorter than other edges.

Ex.: This edge of this solid is shorter than that edge of that one.

6. Find edges of objects in the room and tell whether they are longer or shorter than other edges.

Ex.: This edge of the table is longer than that edge of the desk.

7. Make sentences like this: This edge is longer than that one and shorter than this one.

"Vision and manipulation,—these, in their countless indirect and transfigured forms, are the two coöperating factors in all intellectual progress."—John Fiske.

Relative length.—Scatter sticks of different lengths on a table.

Use one as a standard. Pupils select longer and shorter, and state what they have selected.

After pupil selects a stick and expresses his opinion, let him compare the sticks by placing them together. This will aid him in forming his next judgment.

Select sticks that are a *little* longer or a *little* shorter. This exercise will demand finer discrimination than an

exercise where there is no restriction as to comparative lengths.

Direction and position.—Pupils and teacher point:

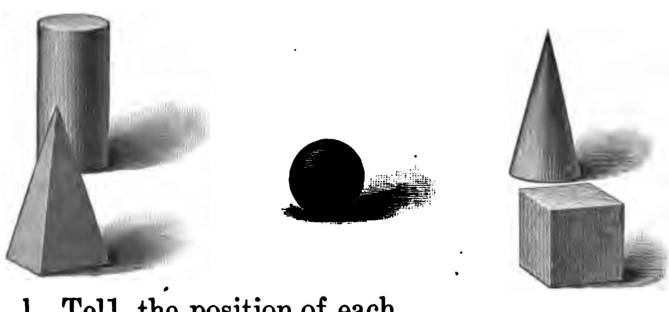
1. Teacher: That is the ceiling. This is the floor. That is the back wall. This is the front wall. This is the right wall. That is the left wall. This is the north wall. That is the south wall. This is the east wall. That is the west wall.

2. A pupil points and teacher tells to what he is pointing. A pupil points and the pupils tell to what he is pointing.

3. Tell the position of objects in the room.

Ex.: There is a picture of a little girl on the north wall. There are three windows in the west wall.

Place groups of solids on three or four desks in different parts of the room, thus:



1. Tell the position of each.

Ex.: The cylinder is at the left at the back. The cube is at the right in front.

2. Without looking tell where the objects are.

Tell where different pupils sit.

Ex. : Mary sits on the second seat in the fourth row from the right.

Place a number of objects on a table.

Let pupils look not longer than ten seconds. Cover the objects. Have pupils tell what they saw. Practise until pupils learn to recognize objects quickly.

Have a pupil from another class walk through the room. Ask pupils to tell what they observed.

Such exercises as the following, if not carried to the point of fatigue, cultivate alertness of mind, concentration, and power to respond quickly to calls for action.

Teacher occupy a pupil's seat, give directions slowly, then place hand where she wishes the pupils to place theirs.

1. Place hand on the front of your desk. On the back. In the middle. At the middle of the right edge. At the middle of the left edge. On the right corner in front. On the left corner at the back. On the left corner in front. On the right corner at the back.

2. Pupil place hand and teacher or other pupil tell where it is.

3. Pupil place an object in different positions on the desk. Pupils tell where it is.

Give each pupil a cube. Teacher use rectangular solid and follow her own directions.

4. Place finger on upper base. On the lower base. On the right face. On the left face. On the front face. On the back face.

5. Pupils place finger and teacher tell where it is placed.

6. Pupils place finger and tell where they have placed it.

Place solids where they can be observed.

"We overlook phenomena whose existence would be patent to us all, had we only grown up to hear it familiarly recognized in speech." — William James.

1. Tell the names of as many as you can.

2. What is the name of the first at the left? Give name if none of the pupils know it. Of the second? Of the third? Of the first, second, and third? Of the fourth? Of the first, second, third, fourth? Of the fifth? Of the five?

3. Look at the solids. Then think of them without looking.

Cover the solids.

4. Give names in order from left to right. From right to left.

5. Tell position.

Ex.: The square prism is the second solid from the left.

Building. — Give pupils a number of cubic inches.

1. Build a prism equal to this one (show prism only for an instant).

Build a prism equal to *this* one.

Build a cube equal to this one.

Give other similar exercises from day to day.

Cutting. — 1. Cut a slip. Cut a longer slip.

2. Cut a slip. Cut a shorter slip.

Give each pupil a square two inches long.

3. Cut larger squares than the square two inches long.

What did I ask you to cut?

4. Cut smaller squares than the square two inches long.

What did I ask you to cut?

5. Cut a square that is neither larger nor smaller than the square two inches long.

Give other exercises.

"Almost invariably children show a strong tendency to cut out things in paper, to make, to build, — a propensity which, if duly encouraged and directed, will not only prepare the way for scientific conceptions, but will develop those powers of manipulation in which most people are most deficient." — Herbert Spencer.

Drawing. — 1. Draw a square. Draw a smaller square.

2. Draw a large square, a small square, and one larger than the small square and smaller than the large square.

3. **Draw two equal squares.**

4. **Draw a line. Draw a longer line.**

5. **Draw a line. Draw a shorter line.**

6. **Draw a line. Draw another neither longer nor shorter than this line. Draw other equal lines.**

Do not push demands in advance of the child's growing power to do.

Through the child's attempts to do that which it wishes, comes the fitting of the muscles for more definite and more complex movements. Above all things let the earlier movements be pleasurable, that an impulse to renewed exertion may be given. The desire to create is the truest stimulus to that action which gives muscular control. Our exactions may make the doing so disagreeable as to destroy the desire to produce.

Relative magnitude. — Place solids where they can be handled.

1. Find solids that are a little larger than other solids.

2. Find solids that are a little smaller.

3. Find objects that are a little larger or a little smaller than other objects.

Ex.: That desk is a little larger than this.

4. Find surfaces of the solids that are a little larger or a little smaller than other surfaces.

5. Find edges of the solids that are a little longer or a little shorter than other edges.

6. Find edges of other objects that are a little longer and those that are a little shorter than other edges.

Cutting. — 1. Cut a slip of paper. Cut another a *little* longer. Another a *little* shorter. Measure. Practise.

2. Cut a square. Cut another a *little* larger. Another a *little* smaller. Measure. Practise.

Drawing. — 1. Draw a line. Draw another a *little* longer. Another a *little* shorter. Measure. Practise.

2. Draw a square. Draw another a *little* longer. Another a *little* smaller. Measure. Practise.

Cutting. — 1. Cut a slip of paper. Try to cut another equal in length to the first. Look at them. Which is the longer? Place them together to see if they are equal. Practise cutting and comparing.

Give each pupil paper and an oblong rectangle.

2. Cut a rectangle as large as, or equal to, the rectangle I have given you. What are you to cut? Is the rectangle you cut as long as the rectangle I gave you? Is it as wide? Does the one you cut exactly cover the one I gave you? Are the two rectangles equal? Practise trying to cut a rectangle exactly the same size as or equal to the one I gave you.

Equality. — "The intuition underlying all quantitative reasoning is that of the equality of two magnitudes." — Herbert Spencer.

1. Find solids and other objects that are equal.

2. Find solids in which the surfaces are all equal.

3. Find solids that have surfaces of only two sizes.

4. Find solids that have surfaces of three sizes.

5. Find solids in which the edges are all equal.

6. Find solids that have edges of two different lengths.

7. Find solids that have edges of three different lengths.

8. Find a solid that has four equal surfaces. How many other equal surfaces has it?

9. Find a solid that has two equal large surfaces.

10. Find a solid that has two equal small surfaces.

11. Find a solid that has four equal long edges.

12. Show me an edge of one solid equal to an edge of another.

13. Show me two edges of a solid which, if put together, will equal one edge of another.

14. Find objects in the room that are equal, or of the same size.

Ex.: Those two windows are equal. Those two erasers are equal.

Give each pupil a square.

1. Cut a square equal to the one I have given you. Compare. Is the square you have cut equal

to the one I gave you? Practise cutting and comparing.

Give each pupil a triangle.

2. Cut a triangle equal to the one I have given you? Compare. Are they equal? Which is the larger?

1. Draw a line. Draw another equal to the first. Measure. Are the lines equal?

Give each pupil a square.

2. Draw a square equal to the one I have given you. Do the squares look exactly alike? Measure. Are they equal?

3. Draw a triangle. Draw an equal triangle. Do the triangles look exactly alike? Are they equal?

1. Show me equal surfaces in the room. Equal edges.

2. Show me the equal long edges of the blackboard. How many equal long edges has the blackboard? How many short? Show me the two equal long edges and the two equal short edges of other surfaces.

3. Show me the two largest surfaces of this box.

4. A chalk-box has surfaces of how many sizes?

Show a real brick or a paper model.

5. How many equal large surfaces has a brick?

How many equal small surfaces? How many other equal surfaces?

6. Show me a surface in one solid equal to a surface in another.

7. Show me two surfaces which, if put together, will equal one surface that you see.

8. Show me one of the longest edges of this box. One of the shortest. One of the other edges.

9. How many equal long edges has the box? How many equal short edges? How many other equal edges?

10. How many rows of desks do you see?

11. Show me two equal rows.

Pupil observe objects. Cover his eyes. Let another pupil substitute an object for one of those observed. Uncover eyes. Pupil tell what was taken away and what was put in its place.

Secure sets of squares and of other rectangles of different dimensions. Scatter sets over the table.[1]

Train pupils to select those that are equal.

Ex.: That square rectangle equals this one, or that oblong rectangle equals this one, or James found a square equal to this one.

Secure variety of statement.

Cutting.— 1. Look at a cube 2 in. long and cut a square equal to one of its surfaces, or look at a

[1] Length of squares,—2 in., 2½ in., 3 in., 3½ in., 4 in. Dimensions of oblong rectangles,—1 × 2, 2 × 2, 3 × 2, 4 × 2, 5 × 2, and others 1 × 3, 2 × 3, 3 × 3, 4 × 3, 5 × 3, 6 × 3.

square rectangle 2 in. long and cut an equal one. What did I ask you to cut?

Let pupils criticise their own work. Do not tell them that the square rectangle they cut is too large or too small; let them compare and tell you. The work will be good, no matter how crude or imperfect, if it is the best the pupil can do. Growth is possible only from the basis of genuine, natural expression.

2. Practise cutting and comparing.

3. Cut a square rectangle two inches long without observing model.

4. Cut a rectangle whose length and width are the same. Measure. Are they equal? What is the name of this figure? Practise.

To-morrow, have pupils cut the square rectangle again.

Have them tell what they cut, in order to learn to associate the language with the thing.

Give pupils square rectangles four inches long and train them to cut, first when observing, then from memory.

Give pupils rectangles 4 in. by 2 in., and tell them to cut rectangles 4 in. by 2 in.

5. What did I tell you to cut? After cutting, compare and measure.

6. What are the names of the three forms that you have cut?

7. What is the width of the square 2 in. long? Of the square 4 in. long?

Why are a child's ideas *necessarily* crude rather than complete? What, then, should be true of his outward representations?

Why is it impossible to secure perfect forms from young children without interfering with mental and moral development?

"We shall not *begin* with a pedantic and tiresome insistence on accuracy (which is not a characteristic of the young mind), but endeavor steadily to lead up to it — to *grow* it — producing at the same time an ever-increasing appreciation of its value." — H. Courthope Bowen.

As before urged, let the work be done freely. Unnatural restraint in expressing results in lack of feeling. It lessens desire to see and to do. The use of things in which mathematical relations are conspicuous furnishes no excuse for disregarding the truth that progress in the power to represent either within or without is ever from the less to the more definite. The child is not troubled by a complexity or a definiteness which it does not see. Teaching in harmony with nature will permit the child to see freely and express freely.

Exercise in judging will gradually increase the power of definite thinking; and exercise in doing the power of definite action.

Drawing. — Draw 6-in. squares on different parts of the blackboard.

Pupils observe and try to draw equal squares. Measure, and try again.

Let one pupil draw and others estimate whether the square is larger, smaller, or equal to the 6-in. square.

Have pupils measure after drawing, so that they may see mistakes and make more accurate estimates.

Draw lines a foot long. Pupils observe the lines and try to draw equal lines.

Let one pupil draw and others estimate whether the lines are longer, shorter, or equal.

Pupils find edges of objects that they think are a foot long.

Without pupils observing you, draw lines a foot long, a little more than a foot long, and others a little less than a foot long.

Arrange obliquely, horizontally, and vertically. Letter *A*, *B*, *C*, etc.

Pupils select different lines. Ex.: The line *C* is less than a foot long. Other pupils tell whether they agree or not.

Have pupils find edges in the room a little more or a little less than a foot long.

Without pupils observing you, draw a line 2 ft. long.

Have pupils estimate the length. Let them measure.

Without pupils observing you, draw lines on the board less than 2 ft., more than 2 ft., and 2 ft. Letter.

Have pupils estimate the lengths. Ex.: I think the line *B* is more than 2 ft. long. Measure.

Have pupils find edges in room a little more or a little less than 2 ft. long.

Draw a 6-in. line on the board. Do not separate into inches. Draw a foot. Pupils look at both lines. How many 6-in. lines in the foot?

Draw a 4-in. line. Pupils observe and draw. Observe the foot and the 4-in. line. How many 4-in. lines in a foot?

Place the solids where they can be handled. Pupils estimate the length of edges. Measure.

Have pupils show edges of solids that they think are 4 in. long.

Have pupils tell how long, wide, and high they think each solid is. Ex.: I think this solid is 4 in. long, 2 in. wide, and 1 in. high, or it is 4 in. by 2 in. by 1 in.

"If the judgment made be original, then the standpoint of the one making the judgment is disclosed." — William T. Harris.

Building. — **If a direction is not understood, the teacher should explain by doing a thing similar to that she wishes done. Thus, if she says build a unit equal to $\frac{2}{2}$ of this one, and the pupils do not understand, she should build a unit equal to $\frac{2}{2}$ of it. Then the pupils should build units equal to $\frac{2}{2}$ of other units.**

1. Using cubes, make a prism equal to this one.
2. Using cubes, make a prism two times as large as this one.

Continue to build prisms two times as large as those selected until this can be done easily.

3. Build a block equal to $\frac{2}{2}$ of this one.
4. Build one equal to $\frac{2}{2}$ of this one. Of *this* one.
5. Build a block equal to $\frac{1}{2}$ of this one. $\frac{1}{2}$ of *this* one.

"Doing, or rather, *expressive* doing, reveals to the teacher the nature of his pupil's knowledge; exhibits to the pupil new connections and suggests others still; develops skill or effectiveness in doing as mere exercise of information seldom does, or does but feebly; and trains the muscles, the nerves, and the organs of sense to be willing, obedient, effective servants of the mind." — H. Courthope Bowen.

Cutting. — Give pupils paper rectangles of different sizes.

1. Cut a rectangle into two equal parts. After cutting, place the parts together to see if they are equal. Practise cutting and comparing the two parts.

2. Cut rectangles into three equal parts. Compare the parts. Are they all equal? Practise.

Drawing. — 1. Draw a line. Place a point in the middle of the line. Measure to see if the parts are equal. Try again. Measure. Is one of the parts longer than the other? Are the parts equal? What is meant by equal? Show me one of the two equal parts. Show me the other.

2. Draw a line. Separate it into two equal parts. Measure. Are the parts equal? Separate the line into four equal parts. Show me one of the four equal parts. Show me three of the four equal parts. Show me the four equal parts.

3. Draw a line. Separate it into three equal parts. Measure. Are the parts equal?

4. Show me where the line should be drawn to separate the blackboard into two equal parts. Point to the two equal parts of the board.

5. Can you see the two equal parts of the floor? Of the top of your desk? Show me two equal parts of other things in the room.

Give each pupil a square.

6. Measure the edges of the square. What is true of the edges of the square? Find other squares in the room.

7. Draw a square. Measure. Are the edges equal? How many equal edges has a square? Practise trying to draw squares.

8. Draw an oblong rectangle. Measure the two long edges. Are they equal? Measure the two short edges. Are they equal? Practise trying to draw oblong rectangles.

Equality. — **Place solids where they can be handled.**

1. Show a part of that solid equal to this one.
2. Show a part of one solid equal to another.
3. Show a part of that rectangle equal to this one.
4. Show other parts that are equal.
5. What part of that solid equals this one? (Give the name of the part if none of the pupils know it.)
6. Show the part of that rectangle equal to this one.
7. What is the name of the part of that rectangle equal to this one?

Building. — Give pupils cubes. Show a unit.

1. Build a unit equal to this one.
2. Separate the unit into two equal parts.
3. This is $\frac{1}{2}$ of the unit.

Show the other half. Hold up the $\frac{2}{2}$.

Put the halves together. Put one half on the top of the other.

Show a larger unit.

4. Build a unit equal to $\frac{1}{2}$ of this one.
5. Build another unit equal to $\frac{2}{2}$ of it.
6. Build another unit equal to $\frac{3}{2}$ of it.

Relative Magnitude. — 1. Draw a line. Separate it into two equal parts. This is $\frac{1}{2}$ of the line. Show me the other half. Show me the $\frac{2}{2}$ of the line.

2. Show me $\frac{1}{2}$ of the top of your desk. Show me $\frac{1}{2}$ of the blackboard. Show me $\frac{1}{2}$ of this solid. Show me $\frac{1}{2}$ of that solid. Show me $\frac{2}{2}$ of that solid.

3. Draw a line. Draw another as long as $\frac{1}{2}$ of the first. Measure.

4. Draw a line. Draw another two times as long. Show me the part of the second line that is as long as the first. What part of the second line equals the first? The first line is as long as what part of the second? The first line equals what part of the second?

5. Cut a slip of paper. Cut another slip $\frac{1}{2}$ as long. Measure. Cut a slip of paper. Cut another equal to $\frac{1}{2}$ of the first. What did I ask you to do?

6. Cut a rectangle. Cut another two times as large. Show me the second rectangle you cut. What part of the second rectangle is as large as the first?

7. Use sticks and lay lines two times as long as other lines.

8. Use sticks and make rectangles two times as large as other rectangles.

Have pupils handle solids and tell into how many equal smaller solids a larger solid can be cut.

Avoid the frequent use of any particular solid, surface, or line, in making comparisons. To use an inch cube, a two-inch cube, a foot, or a yard in the elementary work oftener than other units are used interferes with free mental action.

Place on the table various solids, cardboard rectangles, both square and oblong, and other objects. Let each pupil take one object.

Teacher: John, what have you?
I have a sphere.

Other pupils tell what they have. Pupils tell what other pupils have. Ex.: William has a red square.

Teacher: Who has the largest solid? Who have solids that are alike?

Place objects upon other objects and tell what was done.

Ex.: I put a cone upon a cube. Mary placed a cone upon a cube.

Place two objects together and tell what you did.

Ex.: I put a square and an oblong rectangle upon the table.

Tell what are in a group of three objects.

Ex.: A knife, a pen, and a pencil are in that group. I have a sphere, a prism, and a cylinder.

Relative magnitude. — 1. Tell all you can about *A* and *B*.

2. *B* is as large as how many *A*'s?

3. What part of *B* is as large as *A*? *A* equals what part of *B*?

4. *B* equals how many times *A*?

Place pairs of solids having the ratio two where they can be handled by the class.

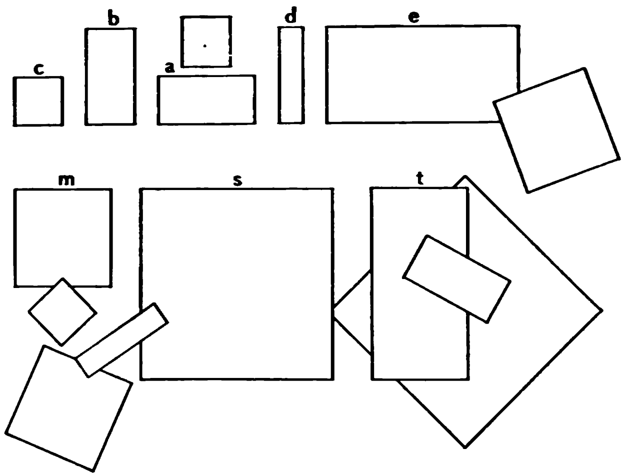
5. Observe solids and make sentences like this: This solid can be cut into two solids each as large as that one.

6. Have pupils discover all the relations they can.

The things between which the relation $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, 2, 3, 4, etc., is seen, should vary. Keep in view the fact that *the thing is its relations.* (See page 19.) That which the pupil sees as $\frac{1}{2}$ when related to a unit twice its size he should see as $\frac{1}{3}$ or 2 according to that with which it is compared. He will do so if there is a proper presentation. At first his perceptions of these relations will be dim. They will gradually develop according to his experience.

"There must be accumulation of experiences, more numerous, more varied, more heterogeneous — there must be a correlative gradual increase of organized faculty." — Herbert Spencer.

"The formation of an idea is an organic evolution which is gradually completed, in consequence of successive experiences of a like kind." — Dr. Maudsley.



Draw the units on the blackboard, making $\dot{C}$ 6 in. long.

1. Tell all you can about these units or rectangles. How many of these rectangles are square? How many oblong?

2. Find the units that are equal.

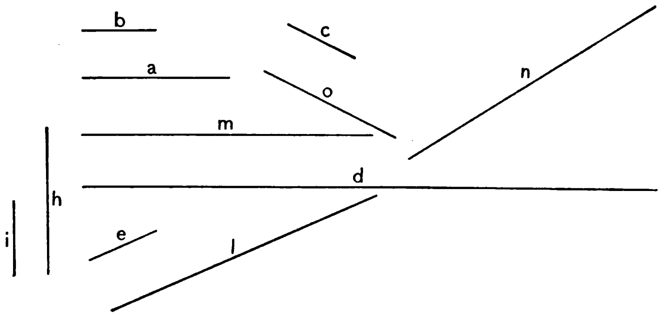
3. The different units can be cut into what? Ex.: The unit *T* can be cut into two *M*'s.

4. Make sentences like this: One half of *B* equals *C*.

5. Find units equal to $\frac{1}{2}$ of other units.

6. Find units two times as large as other units.

7. Draw the units again to a different scale and continue the work.



Draw the units on the blackboard, making *B* 6 in. long.

1. Find out all you can about these units.
2. Find the equal units.
3. Find units equal to $\frac{1}{2}$ of other units.

Ex.: The unit *I* equals $\frac{1}{2}$ of *O*.

4. Make sentences like this: One half of *M* equals *A*.
5. The different units can be made into what units?

Ex.: The unit *M* can be made into four *C*'s.

6. Find units two times as large as other units.

Ex.: The unit *M* equals two times *A*.

The number of repetitions needed will depend greatly upon the manner of presentation. But no art, no mode of work can alter the fact that time is required, that ideas are the result of an organizing process.

Building.—Show a prism 3 by 1 by 1.

1. Build a unit equal to this one.

2. Separate the unit into three equal parts.
3. Show me the three equal parts.
4. Hold up two of the three equal parts.
5. Show me one of the three equal parts.

Show a unit 6 by 1 by 1.

1. Build a unit equal to this one.
2. Separate the unit into three equal parts.
3. Show me the three equal parts.
4. Show me one of the three equal parts.
5. Show me two of the three equal parts.

Show a unit 3 by 2 by 1.

1. Build a unit equal to one of the three equal parts.
2. Build another equal to two of the three equal parts.
3. Build another equal to the three equal parts.

Cutting. — Give each pupil several rectangles of different sizes.

1. Cut a rectangle into three equal parts. What did I tell you to do? Place the three parts together. Are the three parts equal? Practise cutting and comparing.

Drawing. — 1. Draw a line. Separate it into three equal parts. Measure. Is one of the parts shorter than one of the others?

2. Draw lines of different lengths and practise trying to divide them into three equal parts.

3. Draw rectangles of different sizes and practise trying to separate them into three equal parts.

4. Show me where lines should be drawn to separate the blackboard into three equal parts. Move your hand over each of the three equal parts of the blackboard.

Select different solids.

5. Show me where each should be cut to separate it into three equal parts.

6. Find a solid that can be made into three parts, each as large as this solid.

Ex.: That solid can be made into three solids each as large as this one.

Give each pupil a piece of paper on which there is drawn a line equal to *D*.

D

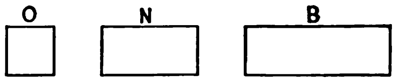
1. Draw a line equal to *D*.
2. Draw a line two times as long as *D*.
3. Draw a line three times as long as *D*.
4. Name the lines *D*, *A*, *B*.
5. *A* is how many times as long as *D*?
6. *B* is how many times as long as *D*?
7. Show me $\frac{1}{2}$ of *A*. *B* is how many times as long as $\frac{1}{2}$ of *A*?
8. Show me $\frac{1}{2}$ of *A*. Draw a line three times as long as $\frac{1}{2}$ of *A*.
9. Draw a line equal to the sum of *D* and *A*.

The sum of *D* and *A* equals what line?

10. If we call *D* 1, what ought we to call *A*? What ought we to call *B*?

11. The sum of *A* and *D* equals what? The sum of 1 and 2 equals what?

Relative magnitude.—Give each pupil a square inch and an oblong 2 in. by 1 in. and another 3 in. by 1 in.



1. What is the length of the square rectangle? How long is the largest rectangle? What is the length of the other rectangle?

2. Show me the rectangle 2 in. by 1 in. The rectangle 3 in. by 1 in. Point to each rectangle and describe it.

Ex.: This is a square rectangle 1 in. long.

3. Call the largest rectangle *B*, the smallest *O*, and the other *N*. Show me *O*. Show me *B*. Show me *N*.

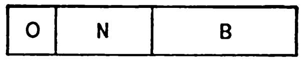
4. *N* is as large as how many *O*'s? What part of *N* equals *O*? *N* equals how many times *O*? *O* equals what part of *N*?

5. *B* is as large as how many *O*'s? *B* equals how many times *O*? Show me $\frac{1}{2}$ of *N*. *B* is how many times as large as $\frac{1}{2}$ of *N*?

6. If we call *O* $\frac{1}{2}$, what is *N*? What is *B*?

7. Cut rectangles equal to *O*, *N*, and *B*.

1. Place *O* and *N* together and make one rectangle of the two. How long is the rectangle you



have made? How wide is it? It is as large as what rectangle? It equals what rectangle?

2. Place *O*, *N*, and *B* together, making one rectangle of the three. How long is the rectangle? This rectangle could be cut into how many *B*'s? Into how many *N*'s?

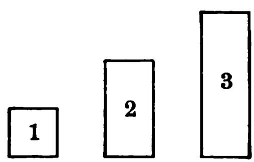
3. Show me $\frac{1}{2}$ of the rectangle. *B* is what part of the rectangle? If you put two rectangles together, the new rectangle is called the sum. The sum of *O* and *N* is what part of the rectangle?

4. If we call *O* 1, what ought we to call *N*? What ought we to call *B*? Show me the unit 3. The unit 2. The unit 1.

Use different magnitudes, and change their arrangement very often. If this is not done the objective representations become the thing, and the *relation*, which is the essence of the subject, is not brought into consciousness at all. We prevent the perception of truth when our presentation limits the relation to particular things. (See preface.)

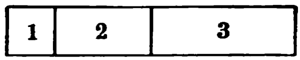
1. Tell all you can about the units 1, 2, and 3.

2. The unit 2 is how many times as large as the unit 1?



What part of 2 is as large as 1? The unit 3 is how many times as large as the unit 1? Show me ½ of the unit 2. The unit 3 is how many times as large as half of the unit 2? The unit 3 is as large as how many halves of 2? The unit 3 equals how many 1's?

1. Place the units 1 and 2 together. The sum



of 1 and 2 equals what? The unit 3 is how much greater than the unit 1?

Ans.: The unit 3 is 2 greater than the unit 1.

The unit 3 is how much greater than 2? How much less is the unit 2 than the unit 3? The unit 1 is how much less than the unit 3? The unit 3 is as large as the sum of what two units? Two and what equal 3? One and what equal 3?

2. Make one rectangle of 1, 2, and 3. The sum of 1 and 2 is what part of the rectangle? The unit 3 is what part of the rectangle?

3. Show me the two equal units that make the rectangle. Show me the three equal units in the rectangle. What three unequal units do you see in the rectangle? Separate the rectangle into two unequal units. What are the names of the two unequal units in the rectangle?

4. The rectangle equals how many 3's? How many 2's?

5. If the 1 is worth a nickel, what is the 2 worth?

6. If you pay a nickel for the 1, how many nickels ought you to pay for the 3?

7. If 2 cost a dime, 1 will cost what part of a dime?

8. The cost of the 1 equals what part of the cost of the 2?

9. The cost of 3 equals how many times the cost of 1?

10. Show me the part of 3 that will cost as much as 2.

11. If an apple costs 3¢, how many 3-¢ will two apples cost?

12. How many times as long will it take to walk two blocks as to walk one block?

13. What part of the time that it takes to walk two blocks will it take to walk one block?

14. If three tops cost 6¢, what part of 6¢ will two tops cost?

Draw the three rectangles on the blackboard to the scale of 1 foot to the inch.

1. If the length of the square rectangle is 1, what is the length of each of the others? What is the height of each?

2. What is the number of feet in the length of each rectangle?

3. Show me the rectangle 1 ft. by 1 ft. The rectangle 1 ft. by 2 ft. The rectangle 1 ft. by 3 ft.

4. Show me the upper edge of the middle rectangle. The lower edge. The right edge. The left edge. How many edges has each rectangle? Show me the entire edge or the perimeter of each.

5. How many feet in the perimeter of the square foot? How many feet in the perimeter of the middle rectangle? In the perimeter of the largest rectangle? Letter the rectangles *O*, *A*, and *B*.

6. Have pupils tell all they can about the relations of the rectangles *O*, *A*, and *B*.

7. Name the rectangles 1, 2, and 3. Have pupils tell all they can about 1, 2, and 3. See questions on 1, 2, and 3 in preceding lesson.

Cutting.—1. Cut a rectangle, making its length and width equal. If we call the length of the rectangle 1, what ought we to call its width? Practise cutting rectangles whose edges are 1 by 1.

2. Cut a rectangle, making its length 2 and its width 1. Measure. The length of the rectangle is how many times its width? If the width of this rectangle is 1, what is its length? Cut rectangles making the dimensions 1 by 3. Measure. Practise.

Drawing.—1. Try to draw rectangles on the blackboard 1 by 1. Measure.

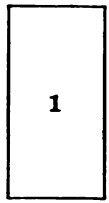
2. Draw rectangles on the blackboard 1 by 2. Measure.

3. Draw rectangles whose edges will be represented by 1 and 2.

4. The length is how many times as great as the height?

Drawing should prolong attention. For the teacher, drawing should be an index of what the child can see and do.

Cutting. — 1. This rectangle is 1. Cut a 1, a 2, and a 3.

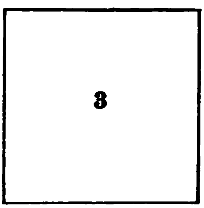


2. The 2 you have cut equals how many times the 1?

3. The 3 you have cut equals how many times the 1?

4. If you put the 1, 2, and 3 together, the sum will make how many 3's? How many 2's? How many 1's?

Drawing. — 1. This is 3. Draw a 3, a 2, a 1.



2. Have a pupil draw a unit on the blackboard and name it 1, 2, or 3. Have other pupils draw the other two units. Use lines and rectangles.

Relative sizes. — Place the cube 1 in. long, the solid 1 in. by 1 in. by 2 in., and the solid 1 in. by 1 in. by 3 in. where they can be seen. Name them *C*, *D*, and *A*.

1. What is the name of the largest unit? The name of the smallest? Of the other unit?

2. Look at the units C, D, and A, and tell all you can about them.

3. D equals how many C's? A equals how many C's? D is how many times as large as C? A is how many times as large as C? A equals how many times C?

4. Show $\frac{1}{2}$ of D. What part of D equals C? A equals how many times $\frac{1}{2}$ of D?

5. Put C and D together. The sum of C and D equals what unit? The sum of C and D equals how many C's? Put C, D, and A together. How many A's in the sum? How many D's? How many C's?

6. If we call C 1, what ought we to call D? What ought we to call A? Show me the 1. The 2. The 3.

7. Look at the units 1, 2, and 3, and tell all you can about them.

8. The unit 2 is as large as how many 1's? The 3 is as large as how many 1's?

9. What part of 2 is as large as 1? Show me the part of 3 that is as large as 1? Show me the part of 3 that is as large as 2.

10. 3 is how much greater than 1? 3 is how much greater than 2? 1 is how much less than 3? 2 is how much less than 3? Put 1 and 2 together. The sum of 1 and 2 equals what unit?[1]

[1] The child understands spoken language before he uses it; he acquires it unconsciously. Let him have the same opportunities

Unite the units 1, 2, and 3. The sum equals how many 3's? How many 2's?

11. If you put *C*, *D*, and *A* together, how high a post will they make? What is $\frac{1}{2}$ of the height of the post? Two inches equals what part of the height of the post? The top of the post is what kind of a rectangle?

1. Cover the eyes of different pupils and place solids in their hands. Let pupils tell relative sizes of solids and surfaces and the relative lengths of edges.

2. Find units that you can call 1, 2, and 3.

3. Show a unit that is two times as large as this one. Show different units that equal two times other units.

Ex.: This unit equals two times that unit.

4. Show different units that equal three times other units.

Ex.: This unit equals three times that one.

5. Tell things like this: This is a 2, for it is two times as large as that unit.

6. Tell things like this: That is a 1, for this unit equals $\frac{1}{2}$ of it.

in learning to associate ideas with sight-forms. From day to day place upon the blackboard the expression for the relations discovered. At first do not ask attention to them. When the child wishes to use them, a great step toward the power to express will have been taken. Let that which you write *mean* something to the child, as that which he hears does; let it symbolize his thought.

7. Find solids whose surfaces represent 1 and 2. How many of the surfaces may we call 1? How many 2?

8. Find surfaces of different solids whose relations are 1, 2, and 3.

9. Find edges that we may call 1 and 2. Tell how many 1's and how many 2's you find in the edges of the solid.

Relations of quart and pint. — Show pupils the pint and quart measures. Have them find the number of pints equal to a quart by measuring.

1. After measuring, tell all you can about the quart and the pint.

This *free* work is far more valuable than that induced by questions. Both the weak and the strong have opportunities to show their power, while the exercise tends to develop self-activity; that is, it fosters a desire to discover when not acting under the stimulus of questions.

Too much questioning interferes with the natural action of the mind in relating and unifying. It isolates ideas. It prevents the teacher from seeing the real state of the pupil's mind. What is wanted is a questioning attitude, — a curiosity which will sustain interest and strengthen attention.

2. What is sold by the pint and by the quart?

3. A quart is how many times as large as a pint?

4. What part of a quart is as large, or as much, as a pint?

5. A quart is how much more than a pint?

6. A pint is how much less than a quart?

7. A quart and a pint equal how many pints?

8. Show me $1\frac{1}{2}$ quarts. What have you shown me?

9. $1\frac{1}{2}$ quarts equal how many pints?

10. If we call a pint 1, what ought we to call a quart? Why?

11. If we call a quart 2, what ought we to call the sum of a quart and a pint?

12. If a quart is 1, what is a pint?

Fill the quart and pint measures with water, and let each pupil lift the two measures.

1. Which is the heavier,—the quart of water or the pint?

2. The quart of water is how many times as much as the pint?

3. What part of the quart weighs as much as the pint?

4. The weight of a pint equals what part of the weight of a quart?

5. The weight of a quart equals the weight of how many pints?

6. A pint of water weighs a pound; how much does a quart of water weigh?

7. What part of a quart of water weighs a pound?

8. The sum of a quart and a pint of water weighs how many pounds?

9. Compare the weight of different solids with the weight of a pint of water.

Ex.: This solid weighs less than a pound, or this solid weighs a little more or a little less than a pound.

10. If a pint of milk costs 3¢, what ought a quart to cost?

11. In a quart there are how many pints? In 3 quarts there are how many 2-pints?

12. How much milk should be put into a quart measure to make it half full?

Relation of the foot and six inches. — Have pupils try to draw lines of the same relative length as the foot and the 6-in. on paper and on the blackboard. After the practice in drawing and in telling what they can about the relations, draw the foot and the 6-in. lines on the board.

1. Tell all you can about these lines.

2. What is the length of the longer line? What is the length of the shorter line?

3. Into how many 6-in. can a foot be separated?

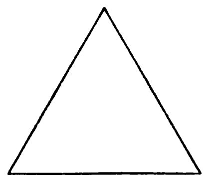
4. A foot is how much longer than 6 inches?

5. 6 in. and how many inches equal a foot?

6. 6 in. are how much shorter than a foot?
7. Show me the part of a foot that equals 6 in.
8. What part of a foot equals 6 in.?
9. A foot is how many times as long as 6 in.?
10. 6 in. equals what part of a foot?
11. How many 6-in. in a foot? In 2 ft.?
12. Two 6-in. equal what?
13. 6 in. and 1 ft. are how many 6-in.?
14. How many 6-in. in $1\frac{1}{2}$ ft.?
15. If we call 6 in. 1, what ought we to call a foot?
16. If a foot is 1, what is 6 in.?
17. If we call 6 in. 1, what ought we to call 2 ft.?
18. Why ought we to call 2 ft. 4, if we call 6 in. 1?
19. Review without observing the foot and the 6-in. line.

Relative length. — Give each pupil an equilateral triangle having a 2-in. base.

1. Cut an equilateral triangle as large as this one. Measure the edges. Are they equal? Practise cutting and measuring.



2. Draw an equilateral triangle. Measure the edges. Are they equal? Practise drawing and measuring.
3. Try to draw a line equal to the sum of two

edges of the triangle. Is the line you have drawn two times as long as one of the edges of the triangle?

4. Draw a line equal to the sum of the edges of the triangle. Is the line you have drawn three times as long as one of the edges of the triangle? Measure. Try again.

5. Show me the perimeter of the triangle. How many 2-in. in the perimeter of the triangle?

6. Let one pupil try to draw an equilateral triangle on the board. Other pupils criticise.

7. Tell all you can about this equilateral triangle.

Relation of the yard and the foot.—Draw a line a

1 YD.

1 FT.

yard long on the blackboard. Draw another a foot long. Give the names of each.

1. What is the name of the longer line? Of the shorter line? Show me the yard.

2. Tell all you can about the yard and the foot.

3. How many feet do you think there are in a yard? Measure. A yard is how much longer than a foot? A foot is how much shorter than a yard?

4. A yard equals how many times a foot?

5. Into how many equal parts must you separate a yard to make each part a foot long?

6. A yard of ribbon contains how many feet?

7. Have pupils try to place points a foot apart on the blackboard. Pupils in class tell whether they are more or less than a foot apart. Measure. Practise.

Estimate the number of yards in different lengths, heights, edges.

Ex.: The height of that door is more than 2 yds. but less than 3.

8. How many feet in a yard? How many 3-ft. in 2 yds.? In 4 yds.?

Problems. — 1. If I have 2 apples in my pocket and $\frac{1}{2}$ as many in my hand, how many have I in my hand?

2. If I pay 4¢ for a yard of ribbon, how much must I pay for $\frac{1}{2}$ yd.?

3. If 1 ft. of molding costs 2¢, how many 2-¢ will 1 yd. cost?

4. If 1 ft. of molding costs 17¢, how many 17-¢ will 1 yd. cost?

5. If $\frac{1}{2}$ barrel of flour lasts 1 month, how long will 1 barrel last?

6. I use 1 yd. of ribbon for a hat and $\frac{2}{3}$ of a yard for a collar; how many feet do I use?

7. I had 4 horses and sold $\frac{1}{2}$ of them; how many did I sell?

8. Mary had a quart of berries and sold a pint. What part of her berries did she sell?

Relative magnitude. — Show pupils 1, 2, 3. Call them $\frac{1}{3}$, $\frac{2}{3}$, 1.

1. What are the names of these units?

2. What is the name of the largest? Of the smallest? Of the next to the largest?

3. Put $\frac{1}{3}$ and $\frac{2}{3}$ together. The sum of $\frac{1}{3}$ and $\frac{2}{3}$ equals what?

4. What must be added to the unit $\frac{2}{3}$ to make the unit 1?

5. The unit 1 is how much larger than the unit $\frac{1}{3}$?

6. You can separate the unit 1 into how many thirds?

7. What part of $\frac{2}{3}$ is as large as $\frac{1}{3}$?

8. What part of 1 equals the $\frac{1}{3}$?

9. What part of 1 equals the $\frac{2}{3}$?

10. The unit 1 is how many times as large as the $\frac{1}{3}$?

11. $\frac{1}{3}$ equals what part of $\frac{2}{3}$? Of 1?

12. Show $\frac{3}{3}$ of the top of this table. Show $\frac{2}{3}$ of it. $\frac{1}{3}$.

Select other solids having the same relative size, and call them $\frac{1}{3}$, $\frac{2}{3}$, 1. Pupils compare. Tell all they can.

1. Show $\frac{3}{3}$, $\frac{2}{3}$, and $\frac{1}{3}$ of different objects in the room.
2. Practise making units of cubes equal to $\frac{3}{3}$ of other units.
3. Practise making units equal to $\frac{2}{3}$ of other units.
4. Practise making units equal to $\frac{1}{3}$ of others.

Give each pupil a square inch, a rectangle 2 in. by 1 in., and one 3 in. by 1 in. Call them $\frac{1}{3}$, $\frac{2}{3}$, 1.

1. Tell all you can about the units $\frac{1}{3}$, $\frac{2}{3}$, and 1.
2. What part of $\frac{2}{3}$ equals the $\frac{1}{3}$?
3. How many $\frac{1}{3}$ in the 1?
4. What part of the 1 equals the $\frac{1}{3}$?
5. The unit 1 is how many times as large as the unit $\frac{1}{3}$?
6. Show me $\frac{1}{2}$ of the $\frac{2}{3}$. The unit 1 is how many times as large as $\frac{1}{2}$ of the $\frac{2}{3}$?

Draw the units on the blackboard to the scale of 1 ft. to the inch.

1. If the largest unit is 1, what is the name of each of the others?

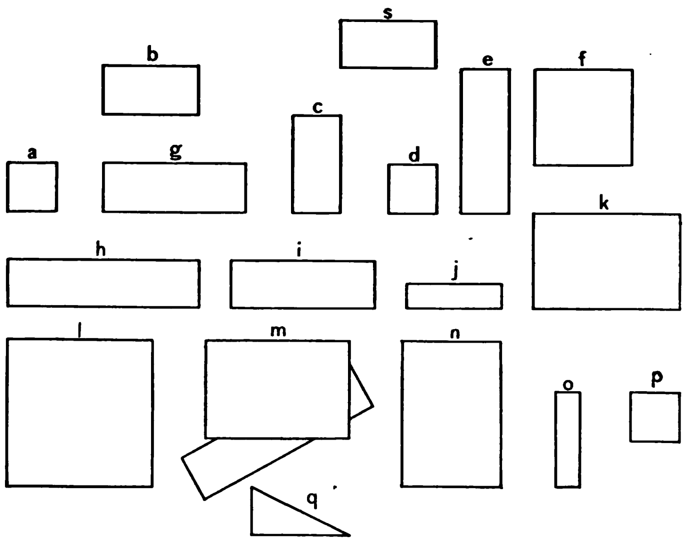
2. Tell all you can about the relations of these units.

Ask questions similar to those above.

3. If the $\frac{1}{3}$ is worth 5¢, what is $\frac{2}{3}$ worth?

4. If the $\frac{1}{5}$ is worth 3¢, how many 3-¢ is the 1 worth?

Draw the figures of the diagram on the blackboard,



making *A* 6 in. long. After pupils have studied and compared them, draw to some other scale.

1. Tell all you can about the relation of these units.

2. If A is 1, how many 1's in the diagram? Can you find five other figures as large as A?

3. If A is 1, how many 2's do you see? How many 3's?

4. If B is 1, how many 1's do you see? How many 2's? How many 3's?

5. If B is 1, how many of the figures are halves?

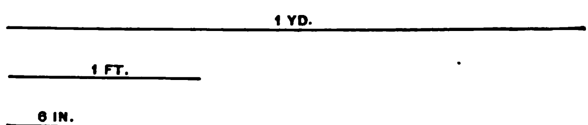
6. If G is 1, how many 1's in the diagram? How many 2's? How many 3's?

7. If G is 1, what is A? If G is 1, how many of the figures are thirds? How many represent $\frac{2}{3}$? How many $\frac{3}{3}$? The figure H equals how many thirds?

8. If A is a 6-in. square, each of the others equals how many 6-in. squares?

9. Make sentences like this: The sum of A and B equals G.

Draw a yard, a foot, and 6 in. on the blackboard.



1. Tell all that you can about the relations of these lines.

2. The yard equals how many feet? The yard is how many times as long as the foot?

3. The foot is how many times as long as the

6-in.? How many 6-in. in the foot? In the yard?

Problems. — 1. The cost of 1 ft. of paving equals what part of the cost of 1 yd.?

2. 1 yd. will cost how many times as much as $\frac{1}{3}$ of a yd.?

3. 1 yd. will cost how many times as much as 1 ft.?

4. The cost of 2 ft. of molding equals what part of the cost of 1 yd.?

5. James has 3 marbles and John has $\frac{2}{3}$ as many; how many has John?

6. If a quart of milk costs 8¢, what part of 8¢ will a pint cost?

7. If a cup of sugar is used in making a cake, how many cups will be needed in making a cake 3 times as large?

8. If the smaller cake is enough for 1 lunch, the larger is enough for how many lunches?

9. If 3 yds. of tape cost 24¢, what part of 24¢ will 2 yds. cost?

10. This line ———————— represents the cost of 1 yd. of cloth; draw a line to represent the cost of $\frac{2}{3}$ of a yd.

11. This line ——— represents the cost of 6 in. of ribbon; draw a line to represent the cost of 1 ft. Of 1 yd.

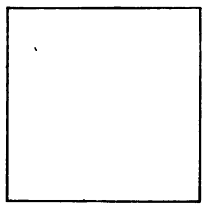
12. If $2 is the cost of $\frac{1}{3}$ of a ton of coal, what

is the cost of 1 ton of coal? Show relative cost by drawing two rectangles.

13. This line ——— represents the cost of 2 ft.; draw a line to represent the cost of 1 yd.

14. 2 ft. of cord cost 6¢. The cost of 1 yd. equals how many halves of 6¢?

Draw a square foot on the blackboard.



1. Show the perimeter of the square foot. What have you shown? How many feet in the perimeter of the square foot?

2. How many 6-in. lines in one edge of the square foot? In the perimeter of the square foot?

Ratios of length. — Draw a foot on the blackboard. Draw a 4-in. line. Pupils practise drawing and measuring these lines.

1. Tell all you can about these lines.

Give the pupils the names of these lines.

2. What is the name of the longer line? What is the name of the shorter line?
3. Into how many 4-in. can a foot be divided?
4. 4 in. and how many 4-in. equal 1 ft.?
5. 2 4-in. and how many inches equal 1 ft.?
6. Show the part of a foot that equals 4 in.
7. What part of a foot equals 4 in.?
8. What part of a foot equals 6 in.?

9. 4 in. equal what part of a foot?

10. A foot is how many times as long as 4 in.? As 6 in.?

11. How many 4-in. in a foot?

12. Show me $\frac{2}{3}$ of a foot. How many 4-in. in $\frac{2}{3}$ of a foot?

13. If we call 4-in. 1, what should we call a foot?

14. If a foot is 3, what is 4 in.?

15. If 4 in. is $\frac{1}{3}$, a foot is how many thirds?

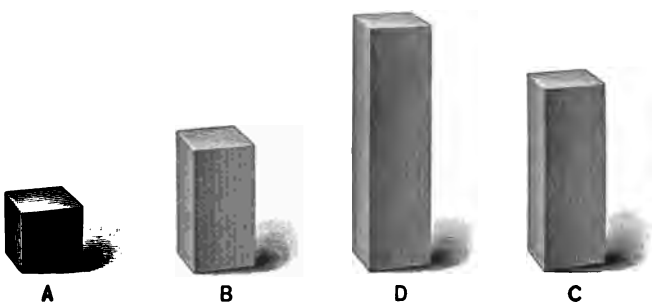
16. Show the part of a foot that is 2 times as long as 4 in.

What part of a foot is 2 times as long as 4 in.?

What part of a foot equals 6 in.?

Review without observing the lines. Have pupils practise placing dots 1 ft. apart. Six inches apart.

Relative size. — Let pupils handle solids which represent 1, 2, 3, and 4. Call them *A*, *B*, *C*, and *D*.



1. What is the name of the largest solid? Of the smallest? Of the next to the largest? Of

the next to the smallest? Give the names in order, beginning with the smallest. What is the name of the unit that is three times as large as A?

2. Tell all you can about the units.

Let it be the constant practice first to permit the pupils to see what they can. The questions of the book are to aid the teacher and not to enslave the pupil. Questions have their value, but when they force details upon a mind unprepared for them, when they destroy the significance of the whole, when they limit individual seeing, when they interfere with the relating, unifying action of the mind, they are intellectual poison.

3. Into how many A's can you divide each unit?

4. Each unit equals how many A's? D equals how many B's?

5. Place A and B together. The sum of A and B equals what unit?

6. Place A and C together. The sum of A and C equals what unit?

7. The sum of A and C equals how many B's?

8. Place B and D together. How many C's in the sum of B and D?

9. The sum of C and B equals how many A's?

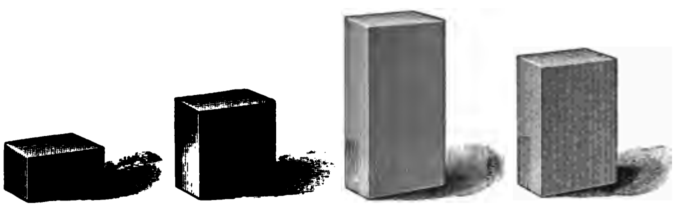
10. The unit B is how many times as large as A? The unit C equals how many times A? The unit D equals how many times B? The unit D equals how many times A?

11. Show me $\frac{2}{3}$ of *C*. The unit *D* is how many times as large as $\frac{2}{3}$ of *C*?

12. What part of *D* equals *A*? What part of *C* equals *A*? Show the part of *D* that is as large as *A*.

13. Show me $\frac{2}{3}$ of *C*. What part of *C* is as large as *B*?

14. If *A* is 2, *B* is how many 2's? *C* is how many 2's? *D* is how many 2's?



A B D C

1. Show me the part of *D* that is as large as *B*. What part of *D* equals *B*?

2. *C* is how many times as large as *A*? Show me $\frac{1}{2}$ of *B*. *C* is how many times as large as $\frac{1}{2}$ of *B*?

3. *D* is how many times as large as *A*? *D* is how many times as large as *B*? Show me $\frac{1}{3}$ of *C*. *D* is how many times as large as $\frac{1}{3}$ of *C*? *D* equals how many thirds of *C*?

4. $\frac{1}{3}$ of *C* equals what unit? $\frac{1}{3}$ of *C* equals what part of *B*? $\frac{1}{3}$ of *C* equals $\frac{1}{4}$ of what unit?

5. Move your finger from the top to the bottom of *A*. Over $\frac{1}{2}$ of *B*. Over $\frac{1}{3}$ of *C*. Over $\frac{1}{4}$ of *D*. What is true of these four units? What units are of the same size as *A*? Show me again the four equal units. What are the names of the four equal units?

6. Move your finger over *B*. Over $\frac{2}{3}$ of *C*. Over $\frac{1}{2}$ of *D*. Show me the three equal units again. What are the names of the three equal units?

7. $\frac{2}{3}$ of *C* equals what unit? $\frac{2}{3}$ of *C* equals what part of *D*?

8. If you cut *D* into 4 equal parts, or into fourths, how many of the fourths will make a unit as large as *C*? $\frac{3}{4}$ of *D* equals what unit?

Use other solids having the same relations as *A*, *B*, *C*, *D*. Give different names to the solids, and review. Then review without solids.

1. If we call *A* 1, what ought we to call *B*? *C*? *D*?

2. If *A* is $\frac{1}{2}$, what is each of the other units?

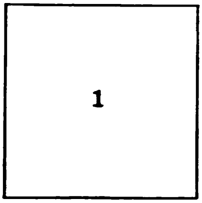
3. If *B* is $\frac{2}{3}$, what is each of the other units?

4. If *A* is 3, how many 3's in each of the other units?

5. If *A* is worth 5¢, how many 5-¢ are each of the other units worth?

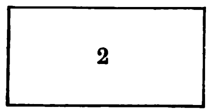
6. If *A* is a box which holds a quart, how many quarts will each of the other boxes hold? How many pints will each box hold?

Cutting. — 1. This is a 1. Cut a 1, a 2, a 3, a 4. You must make the 2 how many times as large as the 1? Have you made the 2 equal to 2 1's? Measure. Have you made the 3 equal to 3 times 1, or 3 times as large as the 1? Measure.



2. How large have you made the 4?

1. This is a 2. Cut a 1, a 2, a 3, a 4. The 1 you cut equals what part of the 2?



2. The 3 you cut is how many times as large as $\frac{1}{2}$ of 2? The 4 you cut is how many times as large as the 2?

Drawing. — Let a pupil draw a unit on the blackboard, and others draw related units and tell what they have drawn.

Relative size. — Place solids having the relation of 1, 2, 3, 4 where they can be handled.

1. If the smallest unit is 1, what is the name of each of the other units? What is the name of the largest unit?

2. Tell all you can about the units.

3. Tell the sums that you see.

Ex.: The sum of 1 and 2 equals 3.

4. Tell how much greater one unit is than another. Ex.: 4 is three greater than 1.

5. 2 and what equal 4? 2 and 2 equal what? 4 equals how many 2's?

6. Put 4 and 2 together. The sum of 4 and 2 can be divided into how many 3's? Into how many 2's?

7. The sum of 4 and 2 is how many times as large as 3? It is how many times as large as 2?

Give each pupil a square rectangle 2 in. long, a rectangle 4 in. by 2 in., a rectangle 6 in. by 2 in., and a rectangle 8 in. by 2 in.

1. Tell all you can about the relations of 1, 2, 3, 4.

2. In each of the rectangles 2, 3, and 4, cover all except the part equal to 1, and tell what part is equal to the 1.

3. Show all the parts that are 2 times as large as 1 and give the name of each.

4. Look for units that are equal to $\frac{1}{2}$ of other units.

5. Estimate the dimensions of each of the rectangles; *i.e.* tell how long and wide you think they are. Measure. State the dimensions. Without observing the rectangles tell the dimensions of each.

1. Place the rectangles 1 and 2 together. The sum of 1 and 2 equals what unit?

2. Show the part of the unit 4 equal to the sum of 1 and 2. What part of 4 equals the sum of 1 and 2?

3. Place the rectangles together and make one rectangle of the four. Show the $\frac{1}{2}$ of this rectangle. The sum of what units makes $\frac{1}{2}$ of the rectangle? What units make the other half? The sum of 2 and 3 equals the sum of what other units?

4. 2 equals what part of 4? How many 4's do you see in the rectangle? Can you find $2\frac{1}{2}$·4's in this rectangle? Show me the 2·4's. Show me the half of 4. Point to the $2\frac{1}{2}$·4's.

5. The large rectangle can be made into how many rectangles as large as 3? Show me the 3·3's. What part of another 3 do you see?

6. Into how many 2's can the rectangle be made?

7. If we should call the square 2, what ought we to call each of the other rectangles? If we should call the square $\frac{1}{2}$, what number of halves would we see in each of the other rectangles? If there are 4 sq. in. in the square rectangle, how many 4-sq.-in. in each of the other rectangles?

Draw the rectangles on the blackboard to the scale of 3 in. to the inch.

8. Tell all you can about the rectangles 1, 2, 3, and 4, drawn on the blackboard.

Problems. — 1. 4¢ will buy how many times as many marbles as 2¢?

2. If you can buy a barrel of flour for $4, how much can you buy for $3?

3. If 1 yd. of cloth costs $3, how much cloth can be bought for $4?

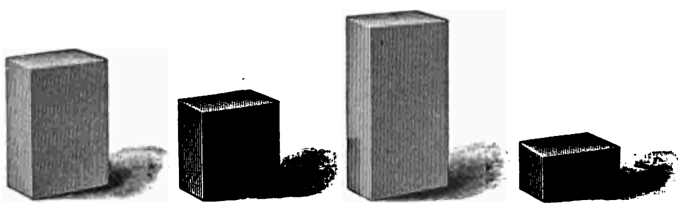
4. If $\frac{1}{4}$ of a basket of fruit is worth 25¢, how many 25-¢ is the basket of fruit worth?

5. If 1 doz. eggs costs 15¢, how many dozen can be bought for 4'15¢?

6. If $\frac{1}{2}$ ton of coal lasts 1 week, how long will 1 ton last?

7. If 1 lb. of butter lasts a family 1 week, what part of a week will $\frac{3}{4}$ of a lb. last?

Show pupils units that represent 1, 2, 3, 4.



B C A D

1. If we call *A* 1, what is *C*? What is *D*? What is *B*?

If pupils cannot give the names ($\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1), tell them.

2. Show me the 1. The $\frac{1}{2}$. The $\frac{3}{4}$. The $\frac{1}{4}$.

3. What is the name of the largest unit? Of

the smallest? Of the next to the smallest? Of the next to the largest?

4. What are the names of these units?

5. Put $\frac{1}{4}$ and $\frac{1}{2}$ together. The sum of $\frac{1}{4}$ and $\frac{1}{2}$ equals what?

6. Put $\frac{1}{4}$ and $\frac{3}{4}$ together. The sum of $\frac{1}{4}$ and $\frac{3}{4}$ equals what?

7. What part of the unit 1 is as large as the $\frac{1}{4}$?

8. What part of the $\frac{1}{2}$ equals the $\frac{1}{4}$?

9. Show the part of the $\frac{3}{4}$ that equals the $\frac{1}{2}$? What part of the $\frac{3}{4}$ equals the $\frac{1}{2}$?

10. $\frac{1}{4}$ and what equal $\frac{3}{4}$?

11. 1 is how much more than $\frac{1}{4}$?

12. $\frac{3}{4}$ and what equal 1?

13. The unit 1 is how many times as large as the $\frac{1}{4}$?

14. The unit 1 is how many times as large as the unit $\frac{1}{2}$?

15. The unit $\frac{1}{2}$ is how many times as large as the unit $\frac{1}{4}$?

16. If the $\frac{1}{4}$ weighs 5 oz., how many 5-oz. do each of the other units weigh?

Review without observing the units.

Building. — 1. Build a unit equal to $\frac{4}{4}$ of this one.

2. Build another equal to $\frac{1}{2}$.

3. Another equal to $\frac{1}{4}$. Another equal to $\frac{3}{4}$.

Show a different unit.

4. Build a unit equal to $\frac{4}{4}$ of this one.
5. Build the $\frac{1}{2}$. Build the $\frac{1}{4}$. Build the $\frac{3}{4}$.

Cutting. — Give each pupil a rectangle. Call it 1.

Cut another 1. Cut $\frac{1}{2}$. Cut $\frac{1}{4}$. Cut $\frac{3}{4}$.

Relation of gallon and quart. — Review lesson on quart and pint. Have pupils practise filling the gallon measure.

Empty it. Fill it $\frac{1}{2}$ full. Empty it. Fill it $\frac{1}{4}$ full. Empty it. Fill it $\frac{3}{4}$ full. After measuring, have pupils tell all they can about the gallon and quart.

1. What is sold by the gallon?
2. A gallon is how many times as much as a quart?
3. What part of a gallon equals 1 qt.?
4. If you should make 2 equal parts of a gallon, how many quarts would there be in each? How many quarts in $\frac{1}{2}$ gal.?
5. A gallon is how much more than a quart?

6. How many quarts must be added to half a gallon to make a gallon?

7. 1 qt. equals what part of a gallon?

8. If we call a quart 1, what ought we to call a gallon? If we call the gallon 1, what ought we to call the quart?

9. A quart equals how many pints? A gallon measure will hold how many quarts? A gallon measure will hold how many 2-pts.?

10. In 3 gals. there are how many 4-qts.?

Relation of gallon, quart, and pint. — 1. If a pint of water weighs 1 lb., how much does a gallon weigh?

2. If a quart of milk costs 6¢, what part of 6¢ will a pint cost?

3. If a quart of milk costs 6¢, how many 6-¢ will a gallon cost?

4. If 1 gal. of milk costs 24¢, what part of 24¢ will 1 qt. cost?

5. If $\frac{1}{4}$ of a gal. of milk costs 6¢, how many 6-¢ will a gallon cost? How many 6-¢ will $\frac{1}{2}$ of a gal. cost? How many 6-¢ will $\frac{3}{4}$ of a gallon cost?

6. The cost of 3 quart boxes of berries is 25¢. The cost of 4 boxes equals how many thirds of 25¢?

Relations of magnitude. — Place units representing 2, 4, 6, 8 where they can be handled. Teach the names 2, 4, 6, 8. If the children know the number relations of 2, 4, 6, and 8, use letters instead of numbers.

1. What is the name of the smallest unit? Of the largest? What is the name of the smaller of the other two? Of the larger? Name the units in order, beginning with the smallest. Name them in order, beginning with the largest.

2. Put two units together and tell what the sum equals.

Ex.: 4 and 2 equals 6.

3. Tell how much less one unit is than another.

4. The sum of 6 and 2 equals what? The sum of 4 and 4 equals what? The sum of 4 and 2 equals what?

5. 4 less 2 equals what? 8 less 4 equals what? 6 less 4 equals what? 8 less 2 equals what? 2 and what equal 4? 4 and what equal 8? 4 and what equal 6? 2 and what equal 8?

6. How many 2's in each unit?

7. Make sentences like this: 2 equals $\frac{1}{2}$ of 4.

8. Make sentences like this: 8 equals 4 times 2.

9. Tell the part of 4, of 6, of 8, that is as large as 2. Tell the part of 6 and of 8 that is as large as 4. Tell the part of 8 that is as large as 6.

10. 4 equals how many times 2? 4 equals what part of 6? Of 8?

11. 6 equals how many times 2? It equals how many times $\frac{1}{2}$ of 4? It equals how many times $\frac{1}{4}$ of 8? 6 is 3 times as large as what unit?

12. 8 equals how many times 2? How many times $\frac{1}{2}$ of 4? How many times $\frac{1}{3}$ of 6?

13. 2 is how many times as large as 1? 4 is how many times as large as 2? 8 is how many times as large as 4?

14. What unit is 2 times as large as 1? As 2? As 4?

1. Find other sets of solids that may be called 2, 4, 6, 8. Tell all the relations that you can.

2. Find surfaces that may be called 2, 4, 6, 8. Tell the relations.

3. Find edges that may be called 2, 4, 6, 8. Tell the relations.

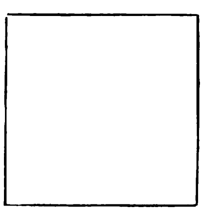
4. Make statements like this: If we call this a 2, we should call this 4, for it is 2 times as large as the 2.

5. Make statements like this: If this is 8, then this is 4, for it equals $\frac{1}{2}$ of 8.

6. Call the blackboard 8. Show the part that is as large as 4. As large as 2. As large as 6.

7. Make statements like this: If we call the edge of this table 2, we must have an edge 2 times as long if we wish to call it 4.

Cutting. — 1. This is a 2. Cut a 2, a 4, a 6, and an 8. Measure to see if you have made each unit the right size.



2. Try again. Cut a large rectangle and call it 2. Cut a 4, a 6, and an 8. Measure.

Drawing. — 1. Draw a rectangle. Call it 2. Draw a 4, a 6, and an 8.

Problems. — 1. If you can clean the blackboard in 8 minutes, what part of the board can you clean in 4 minutes? In 2 minutes? In 6 minutes?

2. The money that you pay for 4 apples equals what part of the money that you pay for 6 apples? For 8 apples?

3. If 2 lbs. of candy cost \$1, how much will 8 lbs. of candy cost?

4. How many times as long will it take to walk 8 miles as to walk 2 miles?

5. If it takes 2 hours to walk 8 miles, how long will it take to walk 4 miles?

6. 6 yds. of ribbon will cost how many times as much as 2 yds.? The cost of 4 yds. equals what part of the cost of 6 yds.?

7. 1 yd. of ribbon will cost how many times as much as $\frac{1}{2}$ yd.? How many times as much as $\frac{1}{4}$ yd.?

8. A gallon measure holds how many times as much as a quart?

9. If a quart of molasses costs a dime, how many dimes will a gallon cost?

10. 6 baskets of apples cost 75¢. What part of 75¢ will 2 baskets cost? What part will 4 baskets cost?

11. 8 hours equals how many thirds of 6 hours? The distance you can walk in 8 hours equals how

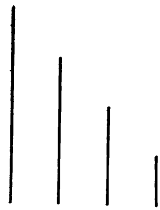
many thirds of the distance you can walk in 6 hours?

Comparing surfaces. — Give each pupil a square 2 in. long, a square 4 in. long, and a rectangle 2 in. by 4 in., and one 2 in. by 6 in., or draw figures on the blackboard of the same relative size.

1. Use the small square as a measure and tell what you can about the relations of the rectangles.
2. Teach the names *A*, *B*, *C*, and *D*. If we call *A* 2, what ought we to call each of the others?
3. Call *A* 1; what is the name of each of the others?
4. Call *A* $\frac{1}{2}$; what ought we to call each of the others?
5. If *A* is $\frac{1}{4}$, how many fourths in each of the others?
6. If *C* is 1, *A* equals what part of 1? *B* equals what part of 1? *D* is how many times as large as the third of 1?
7. Call *D* 1; *B* equals what part of another 1? *A* equals what part? *C* equals what part?
8. If *C* is 3, what is *B*? What is *A*? What is *D*?
9. If you can make 3 1's of *A*, how many 1's can you make of *B*? Of *C*? Of *D*?
10. What is the length of *A*? How many 2-in. in the perimeter of *A*? Of *B*? Of *D*?

Ratios of length. — 1. Practise drawing a foot. Practise placing points 1 ft. apart. Try to draw lines a foot long with eyes closed. Measure. With your eyes closed try to place points 1 ft. apart. Measure.

2. Draw a foot. Draw a line equal to $\frac{1}{2}$ ft. To $\frac{1}{4}$ ft. To $\frac{3}{4}$ ft. Practise drawing and measuring groups of these lines.



3. If we call the shortest line 1, what ought we to call each of the other lines?

4. If the shortest line is $\frac{1}{2}$, what is each of the other lines?

Ans.: 1, $\frac{3}{2}$, 2.

5. If the shortest line is $\frac{1}{4}$, what is each of the other lines?

6. Call the next to the shortest line 1; what is the name of each of the other lines?

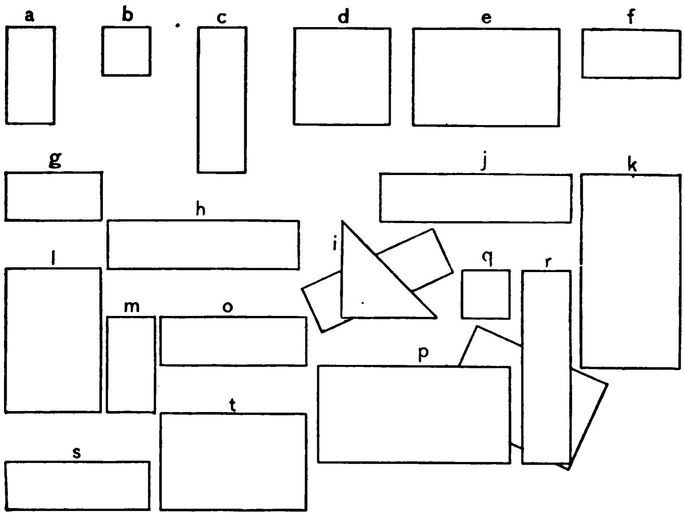
7. If the shortest line is $\frac{1}{3}$, find the 1. If the longest line is 1, find $\frac{1}{4}$.

8. If the longest line is $\frac{1}{2}$, what part of $\frac{1}{2}$ is each of the other lines?

9. Call the longest line 12; what part of 12 is each of the other lines?

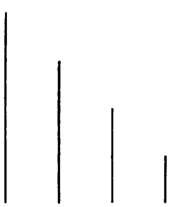
10. Call the shortest line 3; how many 3's in each of the other lines?

Have pupils assign different values to the units and tell what the other units are. Ex.: Call *A* 1; what is each of the others?



Have pupils compare different units with the other units. Ex.: *A* equals 2 times *B*, $\frac{2}{3}$ of *C*, $\frac{1}{2}$ of *D*, etc.

Draw lines on the blackboard 1 ft., 9 in., 6 in., 3 in. long. Teach the names of the lines.



1. What is the name of the longest line? Of the shortest? Of the line that is $\frac{1}{2}$ ft. long? Of the line that is $\frac{3}{4}$ ft. long?

2. Name the lines in order, beginning with the shortest. Repeat. Name in order, beginning with the longest.

3. Make sentences like this: The sum of 6 in. and 3 in. equals 9 in.

4. The 3-in. line equals what part of each of the other lines?

5. The 6-in. line equals how many times the 3-in. line? It equals what part of each of the other lines?

6. Compare the 6-in. line with each of the other lines again.

7. The 9-in. line equals how many times the 3-in. line? It equals how many halves of the 6-in. line? It equals what part of the foot?

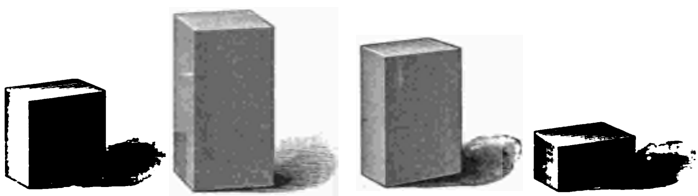
8. The ft. equals how many times the 3-in. line? It equals how many times the 6-in. line? Show $\frac{1}{3}$ of the 9-in. line. The ft. is as long as how many thirds of the 9-in. line? It equals how many thirds of the 9-in. line?

9. A foot is how many times as long as 6 inches? A foot is how much longer than 6 inches? 6 inches equal what part of a foot? 4 inches equal what part of a foot? A foot equals how many 4-inches? What part of a foot is as long as 8 inches?

10. What is $\frac{1}{4}$ ft.? What is $\frac{1}{2}$ ft.? What is $\frac{3}{4}$ ft.? What is $\frac{1}{2}$ of 6 in.? What is $\frac{1}{3}$ of 9 in.? What is $\frac{2}{3}$ of 9 in.? What is $\frac{3}{4}$ of a ft.? Picture the lines in your mind and tell all you can about them.

Review without observing the lines.

Problems.—1. If it takes all your money to pay for a loaf of bread the size of *B*, what part of your money will it take to pay for a loaf the size of *D*? Of *C*? Could you pay for a loaf the size of *A*? The money you have would pay for a loaf three times as large as what part of *A*?



C A B D

2. $\frac{1}{3}$ of *B* is worth a nickel; *B* is worth how many nickels? *A* is worth how many nickels? For *C* you must pay how many times as much as for $\frac{1}{3}$ of *C*? As for $\frac{1}{4}$ of *A*? The cost of *D* equals what part of the cost of each of the other units?

3. Call the blocks cakes. If *C* is enough for six people, *A* is enough for how many people? *D* for what part of twelve people? If *D* is enough for three people, *B* will supply how many?

4. One dollar will buy 8 First Readers. What part of one dollar will pay for 6 First Readers? For 4? For 2?

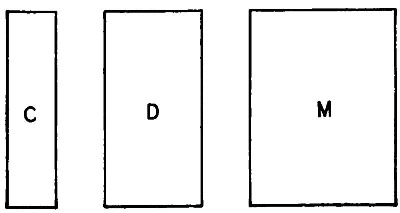
5. A gallon of oil will cost how many times as much as $\frac{1}{4}$ of a gallon?

6. The cost of 2 lbs. of raisins equals what part of the cost of 8 lbs.?

1 doz.	1 doz.
o o o o o o	o o o o
o o o o o o	o o o o
	o o o o

1. How many 6's in a doz.?
2. How many 4's? 3's? 2's?

Give each pupil a rectangle 3 in. by 4 in., another 2 in. by 4 in., and a third 1 in. by 4 in.



1. If *M* is a doz., what part of a doz. is each of the other units?
2. Show $\frac{1}{3}$ of a doz. $\frac{2}{3}$ of a doz. $\frac{3}{3}$ of a doz.
3. A doz. is how many times as large as $\frac{1}{3}$ of a doz.?
4. *M* equals how many halves of *D*?
5. A doz. equals how many halves of $\frac{2}{3}$ of a doz.? Call *C* 4; what is *D*? What is *M*?
6. 4 equals what part of 8? Of a doz.?
7. 8 equals how many times 4? It equals what part of a doz.?
8. A doz. equals how many times 4? It equals how many halves of 8? Which is the more, 4 or $\frac{1}{2}$ of 8?

9. What two equal units in 8? What three equal units in a doz.?

Review, using the rectangles. Review without them.

Problems. — 1. A doz. oranges will cost how many times as much as 6 oranges? As 4?

2. The cost of 9 oranges equals what part of the cost of a doz.?

3. How many 3's in a doz.? How many 4's? How many 6's?

4. If 6 pens cost a dime, how many dimes will a doz. pens cost?

5. One half doz. pears will cost how many times as much as $\frac{1}{4}$ of a doz.?

6. A doz. eggs cost 15¢. 4 eggs cost what part of 15¢? 8 eggs cost what part of 15¢?

7. A doz. bananas will cost how many times as much as 4 bananas?

8. The cost of $\frac{1}{3}$ of a doz. pencils equals what part of the cost of $\frac{2}{3}$ of a doz.?

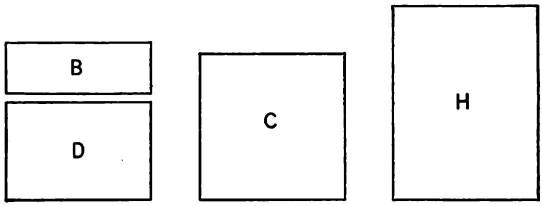
9. The cost of 8 buttons equals what part of the the cost of a doz. buttons?

10. One doz. is how many more than 8? 4 is how many less than one doz.?

11. 6 and what equal a doz.? 8 and what equal a doz.? 4 and what equal a doz.? 3 and what equal a doz.? 9 and what equal a doz.?

Relation of rectangles. — Give each pupil a square rectangle 3 in. long, a rectangle 3 in. by 4 in., a rectangle 2 in. by 3 in., a rectangle 1 in. by 3 in.

1. What are the names of the rectangles in the order of their size?



2. Tell all you can about the rectangles.

3. If *H* represents a doz., what part of a doz. does each of the others represent?

4. The sum of *B* and *D* equals what unit? It equals what part of a doz.?

5. *B* equals what part of each of the other units?

6. What is the relation of *D* to each of the other units? Of *C*? Of *H*? Of the doz.?

7. If *B* is 3, what is each of the other units?

8. How many 3's in 6? In 9? In 12, or a doz.?

9. What is the relation of 3 to each of the other units? Of 6? Of 9? Of 12? Of a doz.?

10. If *H* is worth 10¢, what part of 10¢ is each of the others worth?

11. If *B* cost 5¢, what is the cost of each of the others?

12. If 3 cost 5¢, how many 5¢ will 6 cost? How many 5¢ will 9 cost? 1 doz.?

Review, using the rectangles. Review without them. Drill.

Ratios of time. — 1. How long is it from Christmas to the next Christmas? From one birthday to the next?

2. Draw two lines, one representing a yr., and the other $\frac{1}{2}$ yr., or 6 mos.

3. Tell all you can about the yr. and 6 mos.

1 YR.

6 MOS.

4. How many 6-mos. in a yr.? What part of a yr. equals 6 mos.? 1 yr. and 6 mos. equal how many 6-mos.?

5. How many 6-mos. in $1\frac{1}{2}$ yrs.? 6 mos. equal what part of $1\frac{1}{2}$ yrs.? 1 yr. equals what part of $1\frac{1}{2}$ yrs.? How many 6-mos. in $\frac{3}{2}$ of a yr.?

Draw three lines, one representing a year, one $\frac{1}{3}$ of a year, and the other $\frac{2}{3}$ of a year.

1. If the shortest line represents 4 mos., how many 4-mos. does each of the other lines represent? How many months does each of the other lines represent? How many thirds of a yr. does each of the lines represent?

2. Compare 4 mos. with 8 mos.; with a yr.

3. Compare 8 mos. with 4 mos.; with a yr.

4. Compare 1 yr. with 4 mos.; with 8 mos.

Problems. — 1. The money that Harry can earn in 6 mos. equals what part of the money that he can earn in a yr.? In 8 mos.?

2. The number of months in 1 yr. equals how many times the number in $\frac{1}{2}$ yr.? In $\frac{1}{3}$ yr.?

3. The number of days in 3 mos. equals what part of the number in 4 mos.? In 6 mos.? In 9 mos.? In 1 yr.?

4. One yr. is how much longer than 8 mos.?

5. The time from New Year to New Year equals how many halves of 8 mos.?

Relation of dime and nickel. — 1. If *A* represents a dime, what is the name of the piece of money represented by *B*?

2. How many nickels equal 1 dime?

3. A dime and a nickel equal how many nickels?

4. A nickel equals what part of a dime?

5. The candy you can buy for a nickel equals what part of the candy you can buy for a dime?

6. A nickel equals how many cents?

7. A nickel and how many cents equal a dime?

8. A dime equals how many 5¢?

9. 5¢ equals what part of a dime?

10. A dime and 5¢ equals how many 5¢?

11. $1\frac{1}{2}$ dimes equal how many 5¢?

12. 5¢ equals what part of $1\frac{1}{2}$ dimes?

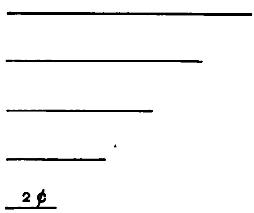
13. A dime equals what part of $1\frac{1}{2}$ dimes?

Relative values. — 1. If the shortest line represents 2¢, what do each of the other lines represent?

2. Point to the different lines and tell what each represents.

3. 2¢ equals what part of 4¢? Of 6¢? Of 8¢? Of a dime?

4. Compare 4¢ with each of the other units. Compare 6¢ with each. Compare 8¢ with each. Compare a dime with each.



Problems. — 1. 2¢ will buy an apple. 4¢ will buy how many apples? How many will 6¢ buy? A dime?

2. A boy sells papers for 2¢ each. How many does he sell to receive a dime?

3. 2¢ is $\frac{1}{3}$ of Nellie's money. How much money has she?

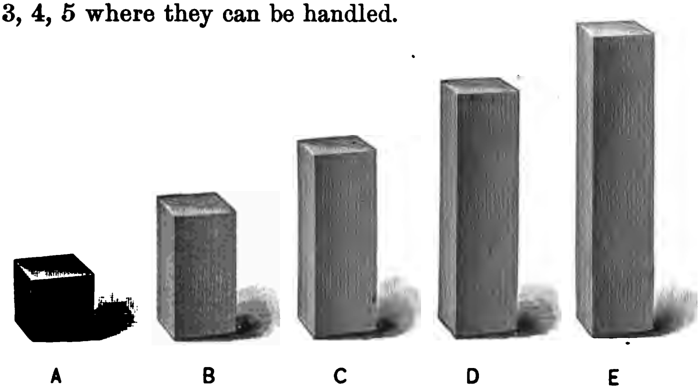
4. John has 10¢ and loses $\frac{1}{5}$ of it; how much does he lose? How many 2¢ has he left?

5. 4 peaches equal what part of 6 peaches? Of 8? Of 10?

6. $\frac{3}{5}$ of a lb. of cheese cost 7¢. How many 7¢ will $\frac{4}{5}$ of a lb. cost?

7. If $\frac{3}{4}$ of a doz. pencils cost 6¢, what is the cost of a doz.?

Ratios of solids. — Place solids which represent 1, 2, 3, 4, 5 where they can be handled.



1. Learn the names *A*, *B*, *C*, *D*, and *E*.

2. Tell all you can about these units. Tell all you can about these units without looking at them.

3. Unite different units and tell what they equal. Ex.: The sum of *A* and *B* equals *C*.

4. Make statements like this: *E* less *A* equals *D*.

5. *B* and what equals *C*? *A* and what equals *C*? *C* and what equals *D*? *B* and what equals *D*?

6. Look at *B*. How many *A*'s equal *B*? What part of *C* equals *B*? What part of *D* equals *B*? What part of *E* equals *B*?

7. *D* equals two times what unit? It is two times as large as what part of *C*? It is two times as large as what part of *E*?

8. What part of *C* is two times as large as *A*? What part of *D* is two times as large as *A*?

9. B equals how many times A? It equals what part of each of the other units?

10. C equals how many times $\frac{1}{4}$ of D? How many times $\frac{1}{2}$ of B? It equals what part of each of the other units?

11. D equals how many times A? D equals how many times B? D is how many times as large as $\frac{2}{3}$ of C? D equals two times what part of E?

1. E equals D and what part of another D? E equals C and what part of another C? E equals how many B's? *Ans.:* E equals $2\frac{1}{2}$ B's. E equals how many A's?

2. Call E 1; each of the other units is what part of another 1?

3. Call D 1; what is each of the other units?

4. Call C 1; what is each of the other units?

5. Call A 1; each of the other units equals how many times 1?

6. Call A 2; what is each of the other units?

7. Call A $\frac{1}{4}$; what is each of the other units? *Ans.:* B is $\frac{1}{2}$, C is $\frac{3}{4}$, D is 1 and E is $1\frac{1}{4}$.

8. Call A $\frac{1}{5}$; what is each of the other units?

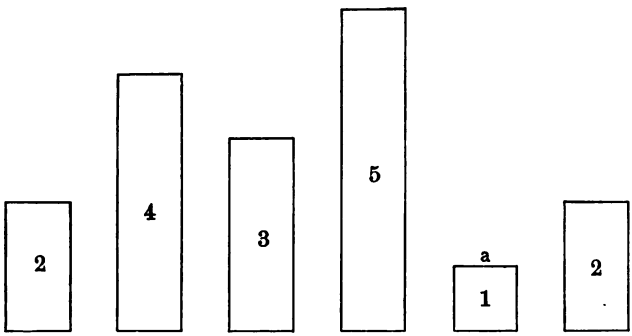
9. Call A $\frac{1}{2}$; what is each of the other units?

10. Call A 3; what is each of the other units?

Drawing and cutting. — 1. Draw a line. Divide it so that one part will represent the unit 2, and the other the unit 3. Measure.

2. Draw a rectangle. Divide it so that one part will represent 2, and the other 3. Measure. Practise.

3. Cut rectangles. Divide them so that they will represent the unit 2 and 3. Measure. Practise. Practise drawing, dividing, and measuring.



Relative areas. — 1. Cut the units 2, 4, 3, 5, 1, 2.

2. Measure[1] each by 2, and tell how many 2's in each.

Ex.: In 5 there are $2\frac{1}{2}$ 2's. In 1 there is $\frac{1}{2}$ of 2.

3. Tell how much more one unit is than another.

Ex.: 4 is three more than 1. 4 is two more than 2.

4. Tell how much less one unit is than another.

5. Unite units and tell what the sum equals. Ex.: The sum of 2 and 1 equals 3.

[1] First, make estimates with the eye. Afterward test judgments by using a measure.

6. What two units equal 4? What other units equal 4?

7. What two units equal 5? What other units equal 5? What three units equal 5?

8. If the unit *A* is $\frac{1}{2}$, each of the other units equals how many halves?

9. Compare each unit with the unit 2.

10. If the unit 2 is worth a dime, what is each of the other units worth?

11. Draw units, making each two times as large as the 2, 4, 3, etc. Measure to see if you have made the units two times as large. Write the names 2, 4, etc.

12. Tell all you can about the units you have drawn.

Review.

"The starting point is, constantly and necessarily, the knowledge of the precise relations, *i.e.* of the equations, between the different magnitudes which are simultaneously considered." — Comte.

1. The unit 3 is how much more than the unit 1? 1 is how much less than 3? 3 apples are how many more than 1 apple?

2. 4 is how much greater than 2? 2 and 2 equal what? 4 is how many times as large as 2? 2 equals what part of 4?

3. 5 is how much greater than 1? Than 3? Than 2? Than 4? What must be added to 3 to make 5? To 1 to make 5? To 2 to make 5?

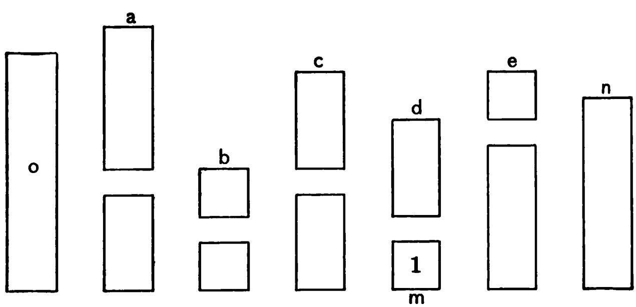
5 pens are how many more than 3 pens? Than 2 pens?

4. The sum of 3 and 2 equals what? Of 1 and 4? Of 2 and 2 and 1? Of 1 and 2 and 2? Of 3 and 2?

5. Henry paid 3¢ for candy and 2¢ for nuts; how much did he pay for both?

6. Nellie spent 5¢ for pears and 2¢ for pins; how much more did she pay for the pears than for the pins?

Separating and combining. — 1. How many 1's do you see in this diagram? How many 2's? How many 3's? How many 4's? 5's?



2. If *d* is 2, what is the name of the units under each letter?

3. Unite the two units under each letter and think the unit to which the sum is equal.

Ex.: Look at the units under *e* and *think 4*.

4. Look at diagram and name sums.

Ex.: Look at the two units under *e* and *say 4*.

5. Draw units on the blackboard and have pupils practise thinking sums.

6. After observing the units carefully, turn away from them and pronounce the sums under each letter.

The expression for quantitative ideas should be acquired as the everyday vocabulary has been, — by repeatedly bringing into consciousness the relations which the terms express.

As the pupil advances, sight forms should suggest ideas, just as spoken words do. But as reading should be approached through sense training, an interest in things, and the power to talk freely, so should the use of written forms in mathematics. At the proper time the teacher should find occasion to present the written expression freely and in such manner that the primary attention is still held to the relations discerned. Gradually the use of language in mathematics should become as automatic as the use of language in other subjects.

The principles which govern practice in aiding a child to think in symbols apply in mathematics as elsewhere. For example, when we wish to acquaint the child with the written symbols for his thought of the color of a black dog, we write, "The dog is black." So, when a pupil tells you that 3 and 2 equal 5, write $\begin{array}{c}3\\ \underline{2}\\ 5\end{array}$ so that his eye may take in the expression as a whole. We should represent the complete, not the partial thought of the pupil, — the equation, not a part of it. Fix the thought so firmly that finally one side of the equation will suggest the other.

Ask the following questions, and write answers on the blackboard: —

3 and 1 equal what?
1 and 2 equal what?
2 and 2 equal what?
2 and 3 equal what?

Answers.

a	*b*	*c*	*d*	*e*
3	1	2	2	1
2	1	2	1	3
5	2	4	3	4

Observe[1] $\begin{array}{c}3\\ \underline{2}\\ 5\end{array}$; image; write.

Observe $\begin{array}{c}1\\ \underline{1}\\ 2\end{array}$; image; write; practise.

Observe $\begin{array}{c}3\\ \underline{2}\\ 5\end{array}\ \begin{array}{c}1\\ \underline{1}\\ 2\end{array}$; image; write.

Observe $\begin{array}{c}2\\ \underline{2}\\ 4\end{array}$; image; write; practise.

Observe $\begin{array}{c}3\\ \underline{2}\\ 5\end{array}\ \begin{array}{c}1\\ \underline{1}\\ 2\end{array}\ \begin{array}{c}2\\ \underline{2}\\ 4\end{array}$; image; write; practise.

Continue adding one combination at a time, until the pupils can image and write the five readily.

Tell the combination under each letter, thus: $\begin{array}{c}3\\ \underline{2}\\ 5\end{array}$ is under *a*.

What combination is under *c*? Under *e*? Under *b*? Show the combination at the left. The second from the right. Image and think each combination with its sum, beginning at the top.

Image each combination and pronounce the sum.

[1] "The habit of hasty and inexact observation is the foundation of the habit of remembering wrongly." — Dr. Maudsley.

"A few such items must be memorized and reviewed daily, adding a small increment to the list as soon as it has become perfectly mastered." — W. T. Harris.

Continue to work with these five combinations until they are indelibly fixed.

Write on blackboard: $\begin{array}{ccccc} 3 & 1 & 2 & 2 & 2 \\ \underline{2} & \underline{2} & \underline{2} & \underline{1} & \underline{3} \end{array}$

Think the sum of each. Pronounce the sum.

Do not say 3 and 1 are 4, nor 3, 1, 4; but observe $\begin{array}{c} 1 \\ \underline{3} \end{array}$ and say 4.

Name sums from right to left, without observing the board. From left to right. What is the second sum from the right? The third from the left? etc.

Make columns of the combinations, omitting sums, thus:

a	*b*	*c*	*d*
3	2	1	1
2	2	1	3
1	3	2	1
1	2	2	1
2	2	1	3
2	1	2	2
2	1	3	2
1	3	1	2
1	1	3	1
3	1	2	2

Have pupils look at each column carefully and image the sum of each combination of two figures. Picture, slowly at first, the combinations under *a*: 5, 2, 4, 3, 4, then more quickly, but not so quickly as to destroy the visual image.

It will be easy to secure rapidity after the habit of imaging has been established.

Image,[1] beginning at the bottom.

Image from right, thus: 4, 2, 4, 5.

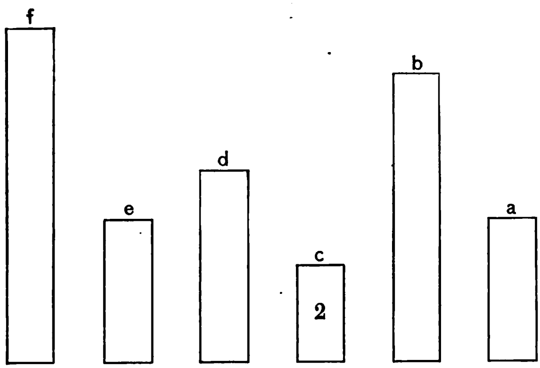
[1] "There can be no doubt as to the utility of the visualizing faculty where it is duly subordinated to the higher intellectual

1. Measure each unit by 2.[1] By 3. By 4.

2. If *c* is 2, what is the name of each of the other units?

3. On each unit that you draw or cut, write the name.

4. Tell all you can about these units.



5. What two units are as large as 6?

6. Into what two equal units can you separate 4?

7. What two equal units in 6? What three equal units in 6? What two unequal units do you see in 6?

operations. A visual image is the most perfect form of mental representation wherever the shape, position, and relations of objects in space are concerned." — Francis Galton.

"Addition, as De Morgan somewhere insisted, is far more swiftly done by the eye alone; the tendency to use mental words should be withstood." — Francis Galton.

[1] Do not permit counting. Wait until the pupil observes and becomes conscious of the relative size of the units.

8. The unit 7 is how much larger than the unit 4? Than the unit 3? 4 is how much less than 7? What must be taken out of 7 to leave 4?

9. 4 and what equal 6? 2 and what equal 6?

10. 4 and what equal 7? 3 and what equal 7?

11. 6 and what equal 7? 2 and what equal 7?

12. 5 and what equal 7?

13. 7 cherries are how many more than 3 cherries? Than 5? Than 2? Than 1?

14. The sum of 3 and 3 equals what? 6 equals how many times 3? 3 equals what part of 6? 6 is how much greater than 3? 3 is how much less than 6?

15. Cora paid 5¢ for paper and 2¢ for a pencil. How much did she pay for both? How much more did she pay for the paper than for the pencil? How much more for both than for the paper? The cost of the pencil equals what part of the cost of the paper?

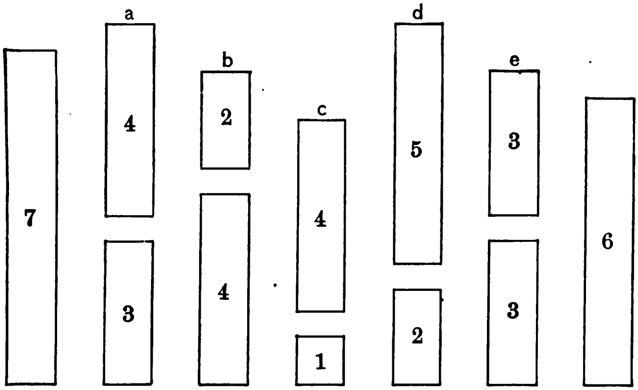
16. If *C* is a rug containing 2 square feet, how many square feet in each of the other rugs?

17. Call *C* 1. What is the name of each of the other units?

18. If the width of *E* is 1 foot, what is its perimeter? How many more feet in the perimeter of *E* than in the perimeter of *C*?

19. Call *C* $\frac{1}{2}$. What is the name of each of the other units?

Draw units on the blackboard.



1. How many 1's do you see in this diagram? How many 2's? How many 3's? How many 4's? How many 5's?

2. Practise looking at the diagram and *thinking* the sums.

3. Look and *name* the sums.

4. Think and name sums without looking at the blackboard.

Ask the following questions, and write answers on the blackboard: —

3 and 3 equal what?
2 and 5 equal what?
4 and 1 equal what?
4 and 2 equal what?
3 and 4 equal what?
(See method of study on pages 125, 126.)

Answers.

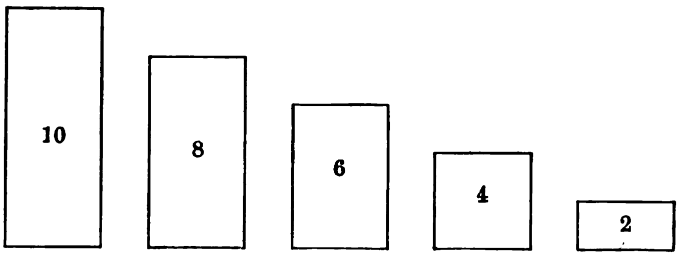
4	2	1	5	3
3	4	4	2	3
7	6	5	7	6

Use any set of solids having the relation of 1, 2, 3, 4, 5.

1. If 2 is the name of the smallest unit, what is the name of each of the others?

2. Give the names beginning with the smallest unit. Give the names beginning with the largest unit.

3. Tell all you can about these units.



4. Unite different units, and tell what the sum equals.

5. Make sentences like this: 8 less 6 equals 2.

6. 4 and what equals 6? 2 and what equals 6? 4 and what equals 8? 6 and what equals 10?

7. Tell what two equal units are found in each unit. Ex.: In the unit 6 there are 2 3's.

8. How many 2's in each unit? Each unit equals how many 2's?

9. What is the relation of 2 to each of the other units?

Use another set of solids having the same relations. Name them 3, 6, 9, 12, 15. Work with these units as you did with the 2, 4, 6, 8, 10.

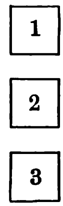
Show a solid. Give it a name, and ask pupils to find a related solid.

Ex.: This is 9; find 3. This is 10; find 2.

Drawing. — Draw rectangles having the relations of 4, 8, 12, 16, 20 on blackboard. Work with these units as you did with 2, 4, 6, 8, and 10.

Draw a rectangle. Give it a name. Pupils draw related rectangles. Ex.: This is a 12; draw a 4.

Draw rectangles either larger or smaller than these above, but having the same relations. Teach these relations through the language 5, 10, 15, etc.



Cutting. — This is a 1. Cut a 1, 2, 3, 4, 5.

This is a 2. Cut a 2, 4, 6, 8, 10.

This is a 3. Cut a 3, 6, 9, 12, 15.

Give other exercises in cutting and drawing, which will fix the relative sizes of these units.

Separating and combining. — 1. Measure each unit by 2.

Ex.: There are $1\frac{1}{2}$ 2's in *O*.

2. How many of the units contain an exact number of the 2's?

3. Measure each unit by 3.

4. If *A* is 2, what is the name of each of the other units?

5. Tell all you can about the relations of these units. Measure each by 2.

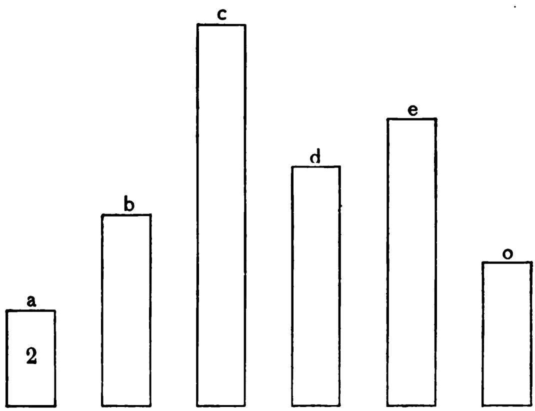
Ex.: There are $1\frac{1}{2}$ 2's in 3.

6. What units united will make 8? What two equal units in 8? What four equal units in 8?

What two unequal units in 8? What other unequal units in 8?

1. The unit 8 is how much larger than the unit 6? Than the unit 5? Than 3? Than 4?

2. How many 4's in 8? 8 is how much more than 4? 8 equals how many times 4? 4 equals



what part of 8? How many 2's in 8? In 6? 6 equals what part of 8?

3. Show me the unit equal to $\frac{3}{4}$ of 8. Show me the unit equal to $\frac{4}{3}$ of 6.

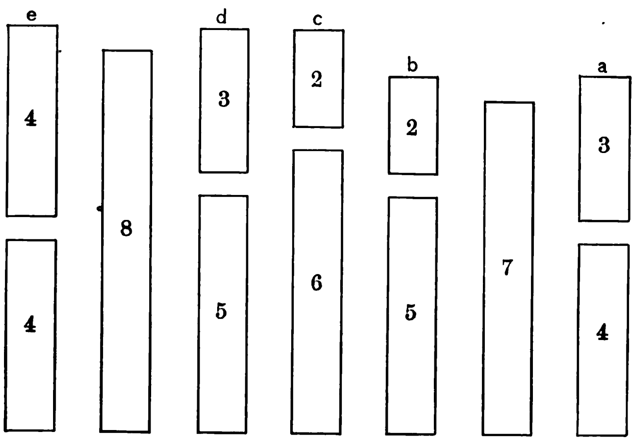
4. What is the sum of 4 and 4? Of 6 and 2? Of 4 and 3? Of 5 and 3? Of 2 and 6? Of 3 and 5?

5. 4 and what equal 8? 6 and what equal 8? 2 and what equal 5? 2 and what equal 8? 5 and what equal 8? 3 and what equal 6? 3 and what equal 5? 3 and what equal 8?

6. A boy had 8 marbles and lost $\frac{1}{2}$ of them. How many had he left?

7. A little girl had 8 dolls. She gave $\frac{3}{4}$ of them to some poor children. How many did she give away?

Draw the units on the blackboard.



Have pupils practise *thinking* units and sums under each letter.

Have pupils *think* and *name* sums without looking at diagram.

Ask the following questions and write answers on the blackboard: —

4 and 3 equal what?
5 and 2 equal what?
6 and 2 equal what?
5 and 3 equal what?
4 and 4 equal what?

Answers.

4	3	2	2	3
4	5	6	5	4
8	8	8	7	7

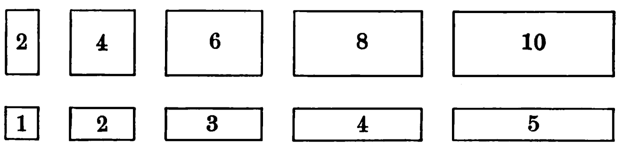
4	3	2	3
4	5	5	5
3	2	3	2
5	6	4	6
4	3	2	2
4	4	5	6
3	4	2	2
4	4	5	6

(See method of study, pages 125, 126.)

Draw units on the blackboard.[1]

1. Draw these units. Write the names: thus, 1, 2; 2, 4; 3, 6; 4, 8; 5, 10.

2. Tell all you can about these units.



3. Make sentences like this: In 6 there are 2·3's.

4. The sum of 4 and 4 equals what? Make sentences like this: The sum of 4 and 4 equals 8.

5. One half of 6 equals what? Make sentences like this: $3 = \frac{6}{2}$. (Read: 3 equals $\frac{1}{2}$ of 6.)

6. The sum of 10 and 5 equals how many 5's?

7. If a 10 and a 5 are put together, the sum equals how many 5's? Make sentences like this: The sum of 10 and 5 equals 3·5's.

[1] In all similar exercises the teacher should draw the units on the board, making them of such size that the eyes of the pupils will not be unduly taxed in observing them.

8. Compare each of the upper units with the one below it.

Ex.: $4 = 2$ times 2.

9. Compare each of the lower units with the one above it.

Tell everything you can about the units without observing them.

Practise thinking the sums of the two units.

Ex.: Look at 1 and 2 and think 3; at 4 and 2 and think 6.

Think sums without observing diagram.

Name sums without observing diagram.

Problems. — 1. If 3 peaches cost a nickel, how many nickels will 6 peaches cost?

2. If 6 peaches cost a dime, what part of a dime will 3 peaches cost?

3. 4 books cost a dollar; what is the cost of 8 books?

4. 10 lbs. of coffee cost how many times as much as 5 lbs.?

5. The cost of 5 lbs. of coffee equals what part of the cost of 10 lbs.?

6. The weight of 3 lbs. of sugar equals what part of the weight of 6 lbs.?

7. The cost of a pt. of milk equals what part of the cost of a qt.?

8. If 2 apples cost 4¢, what part of 4¢ will 1 apple cost?

9. If 2 baskets of apples cost 75¢, what part of the 75¢ will 1 basket cost?

10. If John can walk to school in ½ hour, how long will it take him to walk to school and home again?

Separating and combining. — In this work, aim to secure the association of the three figures in the mental picture. After imaging, test the mental picture by having the pupils supply from memory the figures denoting the sums. Do this in all similar exercises.

$$\begin{array}{rrrr} 2 & 4 & 3 & 5 \\ \underline{2} & \underline{4} & \underline{3} & \underline{5} \\ 4 & 8 & 6 & 10 \end{array}$$

Observe $\begin{array}{r} 2 \\ \underline{2} \\ 4 \end{array}$; image; write. Observe $\begin{array}{r} 4 \\ \underline{4} \\ 8 \end{array}$; image; write; practise.

Observe $\begin{array}{rr} 2 & 4 \\ \underline{2} & \underline{4} \\ 4 & 8 \end{array}$; image; write. Observe $\begin{array}{r} 3 \\ \underline{3} \\ 6 \end{array}$; image; write.

Observe $\begin{array}{rrr} 2 & 4 & 3 \\ \underline{2} & \underline{4} & \underline{3} \\ 4 & 8 & 6 \end{array}$; image; write; practise.

Observe $\begin{array}{r} 5 \\ \underline{5} \\ 10 \end{array}$; image; write; practise.

Observe $\begin{array}{rrrr} 2 & 4 & 3 & 5 \\ \underline{2} & \underline{4} & \underline{3} & \underline{5} \\ 4 & 8 & 6 & 10 \end{array}$; image; write.

Practise until pupils can write the four combinations easily and quickly from memory.

Have pupils practise thinking the sums.

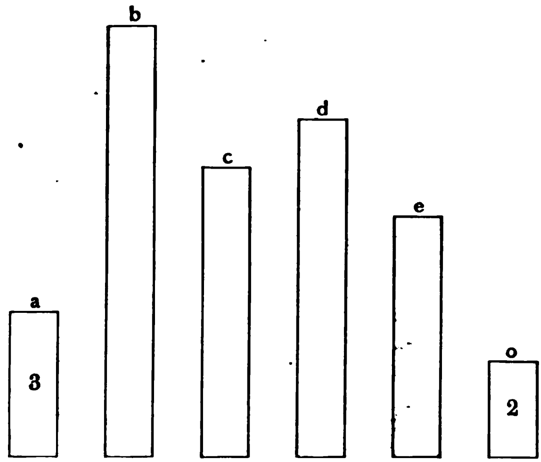
1. Image and pronounce the sums 4, 8, etc.
2. What two equal units in 4? In 8? In 6? In 10?
3. What is $\frac{4}{2}$? (Read: What is $\frac{1}{2}$ of 4?)
4. What is $\frac{6}{2}$? What is $\frac{10}{2}$? What is $\frac{8}{2}$?

   2, 4, 3, 5.
5. Image two of each of the above figures, with the sum. Ex.: Image $\begin{array}{r} 5 \\ \underline{5} \\ 10 \end{array}$. Practise.

Ask pupils the following questions, and write their answers on the blackboard. After having written answers for several days on the blackboard, without calling direct attention to them, see if some of the brighter pupils cannot read the answers. The child should learn the expression without separating it from the thought.

Questions.	*Answers.*
2 = what part of 4?	2 = $\frac{4}{2}$.
3 = what part of 6?	3 = $\frac{6}{2}$.
4 = what part of 8?	4 = $\frac{8}{2}$.
8 = what part of 10?	5 = $\frac{10}{2}$.
2 = how many times 1?	2 = 2 times 1.
4 = how many times 2?	4 = 2 times 2.
6 = how many times 3?	6 = 2 times 3.
8 = how many times 4?	8 = 2 times 4.
10 = how many times 5?	10 = 2 times 5.

Draw on the blackboard.



1. Tell all you can about these units.
2. Measure each by 3. By 2.
3. How many of the units can be exactly measured by 3?
4. The unit *A* is equal to what part of *C*? Of *B*? Of *E*?
5. If *A* is 3, what is the name of each of the other units?
6. Into what equal units can you separate 9?
7. Into what two equal units can you separate 6?
8. What three equal units in 6?
9. What two unequal units in 9? What other unequal units in 9?

1. 9 is how much greater than 6? Than 7? Than 5? Than 4?

2. 5 is how much less than 9? 3 is how much less than 9? 4 is how much less than 9?

3. 3 and what equal 9? 3 and what equal 7? 7 and what equal 9? 5 and what equal 7? 2 and what equal 7? 2 and what equal 5? 6 and what equal 9? 6 and what equal 8?

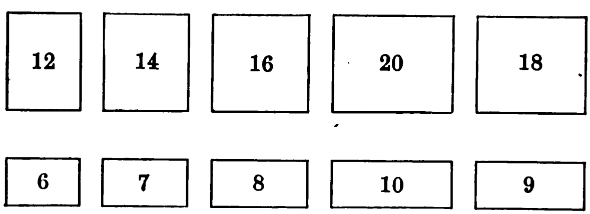
4. How many 3's in 9? 3 equals what part of 9? 6 equals what part of 9? 9 equals how many times 3? 9 is how much more than 3?

5. 9¢ are how many more than 5¢? A nickel and how many cents equal 9¢?

6. A lady can dress 3 dolls in a day. How many can she dress in 3 days?

7. A house has 5 rooms on the first floor and 4 on the second. How many rooms on the two floors?

Ratios. — Draw the units on the blackboard.



1. Draw the units and write the names.
2. Tell all you can about these units.

3. Compare each of the upper units with the one below it.

4. Compare each of the lower units with the one above it.

5. Make sentences like this: The sum of 12 and 6 equals 3 6's.

1. The number of eggs in 1 doz. equals how many times the number in $\frac{1}{2}$ doz.?

2. The number of pts. in 12 qts. equals how many times the number in 6 qts.?

3. The number of pts. in 6 qts. equals what part of the number in 12 qts.?

4. The number of days in 7 wks. equals what part of the number in 14 wks.?

5. The number of cents in 16 nickels equals how many times the number in 8 nickels?

6. The cost of 8 bu. of potatoes equals what part of the cost of 16 bu.?

7. Eggs are 10¢ a doz. How many doz. can be bought for 20¢?

8. 9 ft. equal 3 yds. 18 ft. equal how many 3-yds.?

See work on previous similar table, pages 136, 137.

6	8	7	10	9
6	8	7	10	9
12	16	14	20	18

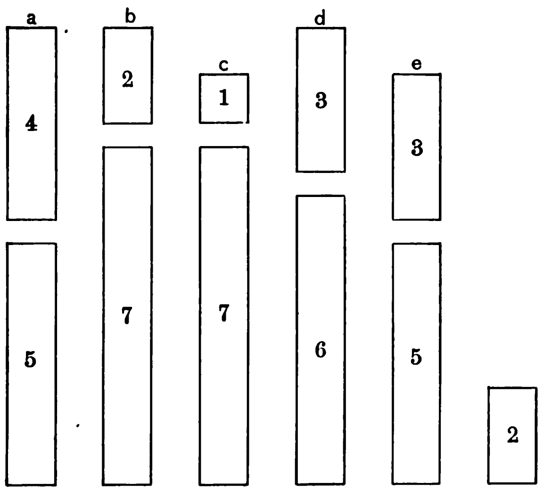
1. What two equal units in 12? In 20? In 14? In 16? In 18?

2. What is $\frac{12}{2}$? (Read: What is $\frac{1}{2}$ of 12?) $\frac{20}{2}$? $\frac{14}{2}$? $\frac{16}{2}$? $\frac{18}{2}$?

6 8 7 10 9

Image two of each of the above figures, with the sum. Ex.: Image $\begin{array}{r}9\\ \underline{9}\\ 18\end{array}$; practise.

Separating and combining. — Draw these units on the blackboard.



1. Point to each and tell its name.
2. What is the sum of the units under each letter?
3. Observe *A* and think 9.
4. Practise imaging *A*, *B*, etc., and think sum.
5. Draw units on blackboard and practise thinking sums.

Ask the following questions, and write answers on the blackboard. **See method of study, pages 125, 126.**

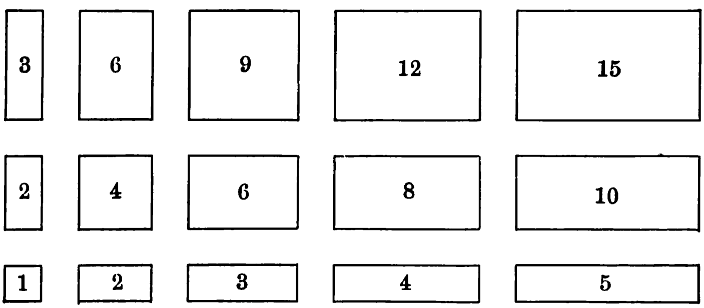
5 and 3 equal what?
6 and 3 equal what?
7 and 1 equal what?
7 and 2 equal what?
5 and 4 equal what?

Answers.

4	2	1	3	3
5	7	7	6	5
9	9	8	9	8

5	1	3	1	7	5	2	3
4	6	6	7	1	4	7	6
3	3	1	5	2	2	4	6
6	6	7	4	7	7	5	1

Ratios. — Draw the units on the blackboard.



1. Draw the units and write their names.
2. Tell all you can about these units.
3. In each set of three units, compare each with the other two. Ex.: $2 = \frac{4}{2}, \frac{6}{3}$.

$4 = 2$ times 2, $\frac{2 \cdot 6}{3}$.

$6 = 3$ times 2, $\frac{3 \cdot 4}{2}$.

4. Tell everything you can about these units without observing them.

1. At 1¢ each, how many postal cards can you buy for 3¢?
2. At 2¢ each, how much will 3 postage stamps cost?
3. If Mary buys 3 rolls at 2¢ each, how much must she pay?
4. If each edge of a triangle is 2 ft. long, how many ft. in the perimeter of the triangle?
5. If a yd. of ribbon costs 3·10¢, how many 10¢ will $\frac{2}{3}$ of a yd. cost?
6. If a lady pays 5¢ for a ft. of picture framing, how much ought she to pay for a yd.?
7. 6 yds. of cloth will make how many times as many doll's dresses as 2 yds.?
8. The cost of the cloth to make 4 dresses equals what part of the cost to make 6 dresses?
9. What is the relation of the cost of 6 yds. to the cost of 4 yds.?
10. 9 books will cost how many times as much as 3 books?
11. The cost of 3 marbles equals what part of the cost of 6 marbles? Of 9 marbles?
12. The number of cents that 12 roses cost equals how many times the number that 4 will cost?
13. There are 8 pts. in 4 qts.; how many 8-pts. in 12 qts.?

14. The number of ft. in 12 yds. equals how many halves of the number in 8 yds.?

15. A string 15 ft. long is how many times as long as a string 5 ft. long?

16. A string 15 ft. long is how many times as long as half of a string 10 ft. long?

$$\begin{array}{cccc} 2 & 4 & 3 & 5 \\ 2 & 4 & 3 & 5 \\ 2 & 4 & 3 & 5 \\ \hline 6 & 12 & 9 & 15 \end{array}$$

Practise until the combinations can be readily written from memory. Try to secure, in each combination, the mental seeing of the three figures and their sum.

This mental habit greatly lessens the labor of learning tables.

What three equal units in 6? In 12? In 9? In 15?

What is $\frac{12}{3}$? (Read: What is $\frac{1}{3}$ of 12?) What is $\frac{6}{3}$? What is $\frac{9}{3}$? What is $\frac{15}{3}$?

4 2 3 5

Image three of each with the sum. Example:

$$\begin{array}{c} 5 \\ 5 \\ 5 \\ \hline 15 \end{array}$$

Image; practise.

Ask questions and write answers of pupils on blackboard.

Questions.		*Answers.*
$2 =$ what part of 4?	Of 6?	$2 = \frac{4}{2}, \frac{6}{3}$.
$3 =$ what part of 6?	Of 9?	$3 = \frac{6}{2}, \frac{9}{3}$.
$4 =$ what part of 8?	Of 12?	$4 = \frac{8}{2}, \frac{12}{3}$.
$5 =$ what part of 10?	Of 15?	$5 = \frac{10}{2}, \frac{15}{3}$.

Questions.

$2 =$ how many times 1?	What part of 3?
$4 =$ how many times 2?	What part of 3?
$6 =$ how many times 3?	What part of 9?
$8 =$ how many times 4?	What part of 12?
$10 =$ how many times 5?	What part of 15?

Answers.

$2 = 2$ times 1, $\frac{2 \cdot 3}{3}$.
$4 = 2$ times 2, $\frac{2 \cdot 6}{3}$.
$6 = 2$ times 3, $\frac{2 \cdot 9}{3}$.
$8 = 2$ times 4, $\frac{2 \cdot 12}{3}$.
$10 = 2$ times 5, $\frac{2 \cdot 15}{3}$.

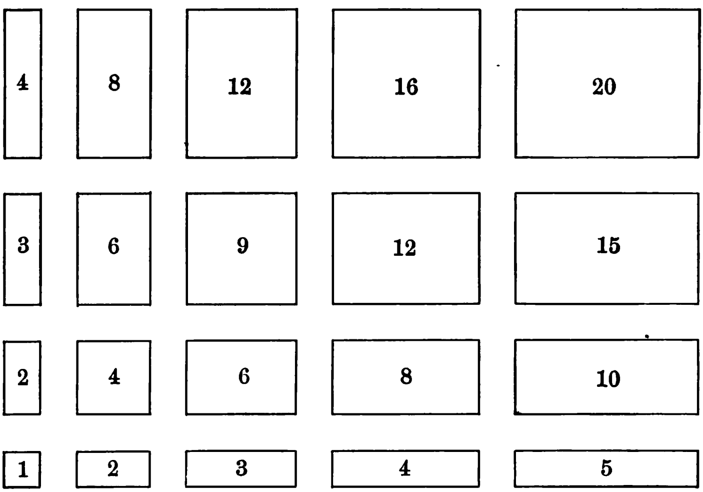
Questions.

$3 =$ how many times 1?	How many halves of 2?
$6 =$ how many times 2?	How many halves of 4?
$9 =$ how many times 3?	How many halves of 6?
$12 =$ how many times 4?	How many halves of 8?
$15 =$ how many times 5?	How many halves of 10?

Answers.

$3 = 3$ times 1, $\frac{3 \cdot 2}{2}$. (Read : $\frac{3}{2}$ of 2.)
$6 = 3$ times 2, $\frac{3 \cdot 4}{2}$.
$9 = 3$ times 3, $\frac{3 \cdot 6}{2}$.
$12 = 3$ times 4, $\frac{3 \cdot 8}{2}$.
$15 = 3$ times 5, $\frac{3 \cdot 10}{2}$.

Ratios. — Draw units on the blackboard.



1. Draw units and write their names; 1, 2, 3, 4; 2, 4, 6, 8; etc.
2. Tell all you can about these units.
3. In each set of four units compare each unit with the other three.

Ex.: $3 = \frac{6}{2}, \frac{9}{3}, \frac{12}{4}$.
$6 = 2$ times 3, $\frac{2 \cdot 9}{3}, \frac{12}{2}$.
$9 = 3$ times 3, $\frac{3 \cdot 6}{2}, \frac{3 \cdot 12}{4}$.
$12 = 4$ times 3, 2 times 6, $\frac{4 \cdot 9}{3}$.

1. If 4 tops cost 20¢, what part of 20¢ will 2 tops cost? One top? 3 tops?

2. 3 hats cost $12; what is the cost of 1 hat? Of 2 hats? Of 4 hats?

3. 2 doz. buttons cost 3 dimes; what is the cost of 4 doz.? Of 1 doz.? Of 3 doz.?

4. 12 lbs. of butter cost $2; what part of $2 will 3 lbs. cost? What will 6 lbs. cost? What part of $2 will 9 lbs. cost?

5. Call 6 $\frac{2}{3}$; what is 12? What is 3? What is 9?

6. 9 boxes of strawberries cost 75¢; what part of 75¢ do 6 boxes cost? 3 boxes? 12 boxes?

7. 16 color boxes cost a certain sum; what part of the sum will 4 cost? 8? 12?

8. What is the relation of 4 to 12? Of 8 to 12? Of 16 to 12?

9. A doz. cost a dime; what is the cost of 4? Of 8? Of 16?

10. There are 20 things in a score; 5 equals what part of a score? 10 equals what part? 15 equals what part? In 5 score there are how many 20's?

11. 5 is $\frac{1}{4}$ of what unit? 10 equals what part

of the unit? 15 equals how many 4ths of the unit?

Ratios. — (*d*)

2	4	3	5
2	4	3	5
2	4	3	5
2	4	3	5
8	16	12	20

1. Learn (*d*) as the other tables have been learned.

2. Compare each unit with the other three; thus: $2 = \frac{4}{2}, \frac{6}{3}, \frac{8}{4}$.
$4 = 2{\cdot}2, \frac{2\cdot6}{3}$ (read, $\frac{2}{3}$ of 6), $\frac{8}{2}$.
$6 = 3{\cdot}2, \frac{3\cdot4}{2}, \frac{3\cdot8}{4}$.
$8 = 4{\cdot}2, 2{\cdot}4, \frac{4\cdot6}{3}$.

3. What four equal units in 12? In 8? In 16? In 20?

4. What is $\frac{8}{2}$? $\frac{12}{2}$? $\frac{16}{2}$? $\frac{20}{2}$?

5. What is $\frac{8}{4}$? $\frac{12}{4}$? $\frac{16}{4}$? $\frac{20}{4}$?
2 4 3 5

6. Image four of each of the above figures, with the sum. Ex.: Image $\begin{array}{r}5\\5\\5\\5\\\hline 20\end{array}$; practise.

Ask questions, and write pupils' answers on the blackboard.

Questions.

2 = what part of 4 ?	Of 6 ?	Of 8 ?
3 = what part of 6 ?	Of 9 ?	Of 12 ?
4 = what part of 8 ?	Of 12 ?	Of 16 ?
5 = what part of 10 ?	Of 15 ?	Of 20 ?

Answers.

$2 = \frac{4}{2}, \frac{6}{3}, \frac{8}{4}.$
$3 = \frac{6}{2}, \frac{9}{3}, \frac{12}{4}.$
$4 = \frac{8}{2}, \frac{12}{3}, \frac{16}{4}.$
$5 = \frac{10}{2}, \frac{15}{3}, \frac{20}{4}.$

Questions.

2 = how many times 1 ?	What part of 3 ?	Of 4 ?
4 = how many times 2 ?	What part of 6 ?	Of 8 ?
6 = how many times 3 ?	What part of 9 ?	Of 12 ?
8 = how many times 4 ?	What part of 12 ?	Of 16 ?
10 = how many times 5 ?	What part of 15 ?	Of 20 ?

Answers.

$2 = 2$ times 1, $\frac{2\cdot 3}{3}, \frac{4}{2}.$
$4 = 2$ times 2, $\frac{2\cdot 6}{3}, \frac{8}{2}.$
$6 = 2$ times 3, $\frac{2\cdot 9}{3}, \frac{12}{2}.$
$8 = 2$ times 4, $\frac{2\cdot 12}{3}, \frac{16}{2}.$
$10 = 2$ times 5, $\frac{2\cdot 15}{3}, \frac{20}{2}.$

1. What is $\frac{12}{2}$? Of 8 ? Of 4 ?
2. What is $\frac{9}{3}$? Of 12 ? Of 6 ?

3. What are $\frac{2 \cdot 12}{3}$? Of 9? Of 6?

4. 5 equals what part of 10? Of 20? Of 15?

5. What is the relation of 10 to 20? Of $\frac{10}{2}$ to $\frac{20}{2}$? Of $\frac{10}{5}$ to $\frac{20}{5}$?

6. 2·3's equal what part of 9? Of 12?

7. 6 equals $\frac{2}{3}$ of what? 6 equals $\frac{3}{2}$ of what?

8. 16 equals how many times 8? How many times $\frac{8}{2}$?

9. What is the relation of 12 to $\frac{8}{2}$?

Problems. — 1. A boat sails 4 miles in $\frac{1}{2}$ hr.; how far does it sail in 1 hr.? In $1\frac{1}{2}$ hrs.?

2. James is 5 yrs. old. His age equals $\frac{1}{3}$ of his brother's; how old is his brother?

3. $10 is $\frac{2}{3}$ of my money, what is $\frac{1}{3}$?

4. If 1 apple costs 3¢, how many apples can be bought for 9¢? For 12¢?

5. 3 bonnets cost $9, how many bonnets can be bought for $6?

6. If a family uses 12 loaves of bread in 1 week, what part of a week will 9 loaves last?

7. I paid $\frac{1}{3}$ of my money for coal and the rest for flour; what part of my money did I pay for the flour?

8. If 2 horses eat a bushel of oats in a day, how much do 3 horses eat?

9. If 3 girls sweep the floor in 10 min., what part of the floor will 2 girls sweep in the same time?

10. 3 oranges cost 5¢, how many 5¢ will 12 oranges cost? A doz.?

11. John has 6¢, and his brother has 2 times as many; how many has his brother? The two boys have how many 6¢?

12. 3 pairs of shoes cost $9, how many $9 will 6 pairs cost?

13. If it takes 15 boys one day to dig a ditch, what part of the ditch can 5 boys dig in 1 day? How many days will it take the 5 boys to dig the other $\frac{2}{3}$ of the ditch?

14. At $$\frac{1}{2}$ a bushel, how many bushels of apples can be bought for $3?

15. Mary is 5 yrs. old. Jane is 4 times as old. How old is Jane?

16. Roy walks 2 blocks, while his sister walks 1 block. Roy walks how many times as fast as his sister? If Roy can walk to school in 4 min., it will take his sister how many 4-min.? How many min.? If John walks 3 times as fast as Roy, John will walk how far, while Roy is walking 2 blocks?

17. Draw a line and call it the distance Roy walks in 1 min. Draw another, showing how far his sister walks in the same time. Draw one showing how far John walks in the same time?

18. Caroline has 8 roses; to how many little girls can she give 2 and yet keep 2 herself?

19. Nettie and Addie are in the middle of the

room. If Addie walks 3 yds. north, and Nettie 2 yds. south, how far apart will they be?

20. A boy had 20¢, and lost 16¢; what part of his money did he lose?

21. Mr. Jones lives 4 blocks east of the schoolhouse, and Mr. Brown 3 blocks west; how far apart do they live?

22. Howard and Frank bought a box of marbles for 6¢. Howard paid 4¢, and Frank 2¢; what part of the marbles ought each to have?

23. A boy sells papers at 2¢ each; how many does he sell to receive 10¢? To receive 8¢?

24. If he sells $\frac{1}{5}$ of 10 papers, he will receive how many cents? If he sells $\frac{3}{5}$ of 10 papers?

25. A lady gave to Carrie 6 apples and to Fannie $\frac{3}{4}$ of 8 apples; to which did she give the greater number?

26. 4 peaches equal what part of 6 peaches? Of 8 peaches? Of 10 peaches?

27. The cost of 6 peaches equals what part of the cost of 8? Of 10?

28. The jelly that 4 peaches will make equals what part of the jelly that 6 peaches will make? That 10 will make? That 8 will make?

29. How many pts. equal 1 qt.? A gallon equals how many qts.? A year equals how many 6-mos.? How many 4-mos.? How many 3-mos.? How many 6's in a dozen? How many 4's? How many 3's?

30. A dime equals how many nickels? 2 dimes equal how many nickels?

31. 1 yd. equals how many ft.? 5 yds. equal how many 3-ft.? How many yds. in 6 ft.? In 9 ft.? In 12 ft.?

32. If a yd. of cord is worth 15¢, what are 3 ft. of cord worth?

33. In a score there are how many 10's? How many 5's? 10 equals what part of a score? 15 equals what part of a score?

34. There are 7 days in a week; how many 7-days in 9 weeks? 14 days equal how many weeks? 7 days equal what part of 3 weeks?

35. There are 5 school days in a week; how many school days in 3 weeks?

36. 9 tons of coal last a family 6 mos.; how many 9-tons will last a yr.? How many tons?

37. If 2 barrels of flour last 4 mos., how many 2-barrels will last a yr.? How many barrels?

38. If you pour a pint of milk into a qt. measure, it fills what part of it?

39. Mr. Robinson sells 2 pts. of milk for 6¢; how much ought he to receive for a qt.?

40. 2 qts. of water fill what part of a gallon measure? 3 qts. of water fill what part of a gallon measure?

41. If you take a qt. of milk out of a gallon of milk, what part of a gallon remains?

42. From a piece of cloth 20 yds. long a